University of Canterbury

Ph.D Thesis

Relativistic Physics in the Clifford Algebra $C\ell(1,3)$

by

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Abstract

There is growing evidence that the Clifford algebra $C\ell(1,3)$ is the appropriate mathematical structure to formulate physical theories. The geometries of 3-space and spacetime are naturally reflected in the algebras $C\ell(0,3)$ and $C\ell(1,3)$ respectively. The choice of metric is important and we give further evidence that only the anti-Euclidean metric allows a proper treatment of rotations.

The algebra $C\ell(1,3)$ is not a division algebra. The invertibility or non-invertibility of elements in the algebra gives physical insight into the limitations of physical systems and non-invertbility should therefore not be regarded as a weakness of the algebra.

The Lorentz force law is shown to arise from energy considerations of the electromagnetic field. This result shows that the Lorentz force is not a necessary addition to Maxwell's equations but rather follows from supplementing the electromagnetic energy density by Hamilton's principle.

Maxwell's equations are written as a single geometric equations in $C\ell(1,3)$. We review this derivation and other electromagnetic theory in the Clifford algebra framework. Taking the massless limit of Weinberg's spin one field equations results in a set of equations more general than Maxwell's equations, containing extra scalar fields. A derivation of these equations in $C\ell(1,3)$ is presented and it is shown that, like the Maxwell equations, this set of equations can also be written as a single geometric equation.

It has been suggested that the stabilised Poincaré-Heisenberg algebra gives an algebraic signature of quantum cosmology. It is shown that there exists a limit in which this algebra reduces to the conformal algebra. This limit describes how the present day Poincaré-algebraic description relates to the conformal-algebraic description of the universe in the past. Furthermore, the proposed algebra inevitably leads to geometric changes in the underlying physical space and any cosmologically derived quantum effects may carry a strong polarisation and spin dependence. The algebra introduces a new dimensionless parameter, the importance of which has been difficult to pin down in the past. It is shown that this dimensionless parameter is closely related to the geometry of the underlying space and if non-zero will affect some of the quantum relativistic notions.

The non-scalar basis elements of $C\ell(1,3)$ are shown to generate the stabilised Poincaré-Heisenberg algebra under the Lie bracket $[x,y]=xy-yx$. The advantage of the $C\ell(1,3)$ approach to the stabilised Poincaré-Heisenberg algebra is that it avoids the traditional stability considerations. It has been previously noted that gravitational effects in quantum measurement necessarily renders spacetime non-commutative and induces modifications

to the fundamental commutators. This non-commutativity of spacetime and the corresponding modifications to the fundamental commutators arise naturally from the algebra $C\ell(1,3)$.

The study of the conformal group in $\mathbb{R}^{p,q}$ usually involves the conformal compactification of $\mathbb{R}^{p,q}$. This allows the transformations to be represented by linear transformations in $\mathbb{R}^{p+1,q+1}$. This embedding into a higher dimensional space comes at the expense of the geometric properties of the transformations. We show that this linearization procedure can be achieved with no loss of geometric insight, if, instead of using this compactification, we let the conformal transformations act on two copies of the associated Clifford algebra.

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Chapter 1

Introduction

1.1 Motivation and Aims

Books by Penrose [1], Smolin [2] and Woit [3] make a plea for a renewed attempt at getting the foundations of physics right. In these texts it is suggested that there need to be changes to at least one, but more probably several of the current understandings of special relativity, quantum mechanics, quantum field theory, the standard model of particle physics and general relativity.

Since my honours project four years ago, discussions and research with Philip Butler and Peter Renaud at the University of Canterbury have led to an increased understanding of the foundational issues involved. Repeatedly we have found that it is most important to have the physics and the mathematics that we use to describe the physics firmly linked to the geometry of space or spacetime. The mathematical structure that arises from the geometry of the underlying space (spacetime) is that of Clifford algebras.

As our understanding of geometry, Clifford algebra and how Clifford algebra may be used to formulate physics increases, we continue to find it a most valuable and versatile tool for formulating physical theories that are rich in geometry. Many of the chapters of this thesis serve to highlight this. We investigate how Clifford algebra can be used to formulate a geometrically rich theory of electromagnetism and attention is given to the role Clifford algebra plays at the interface of the quantum and relativistic realms where the Heisenberg and Poincaré algebras are combined into one larger algebra. The Clifford algebra $C\ell(1,3)$ gives geometric insight into the nature of the underlying physical space in this physical realm.

With the exception of chapters 5 and 7, all the chapters in this thesis rely heavily on the use of Clifford algebra, particularly the algebra $C\ell(1,3)$. For this reason, the second chapter of this thesis is devoted to providing justification for using Clifford algebra and showing how the geometry of space and spacetime is naturally reflected in the Clifford algebras $C\ell(0,3)$ and $C\ell(1,3)$ respectively. This gives us confidence that Clifford algebra is a suitable mathematical structure for formulating physical theories.

We have found that to get the foundations of physics right, the first changes we wish to make to the physics, and to the mathematics we use to describe the physics, are changes at the geometric foundations. We show in chapter 5 that the Lorentz force law may be derived from nothing more than the electric and magnetic fields satisfying Maxwell's equations and conservation of energy. A consequence of this is that electromagnetic fields interact with other electromagnetic fields.

The material and topics covered in this thesis are at different stages of completeness. Some of the material has been published [4–6], other material is being submitted for publication [7–9] and some of the material, chapters 10.1 and 6, needs to be developed more before it is ready for publication. More detail is given in section 1.3.

1.2 Outline of Thesis

There is growing evidence that the appropriate mathematical structure to formulate physics is the $C\ell(1,3)$. The vector space associated with this algebra is sixteen dimensional. This algebra can be derived from the underlying geometry of space-time. In chapter 2 of this thesis, we summarise the attempt made in [8] to show how the Clifford algebra $C\ell(1,3)$ may be arrived at starting from the underlying geometry of space and spacetime. The assumptions made about space and time are carefully considered. An attempt to enumerate these assumptions leads, via the homogeneity and isotropy of space, to the use of Clifford algebras. It is demonstrated that the usual vector algebra with polar and axial vectors, and dot and cross products, needs several small but important adjustments if it is to include all we know about the isotropy and homogeneity of space. These adjustments lead to the Clifford algebra $C\ell(0,3)$ to describe the geometry of 3-space. It is commonly believed that the choice of metric for space or spacetime is not important and merely a matter of taste. We will show that the choice of metric is in fact very important and only a anti-Euclidean metric with signature $(-,-,-)$ allows for a proper treatment of rotations.

Matrices are a natural and useful way of studying the properties of algebras. In chapter 3, the matrix representations of various Clifford algebras are presented and discussed. The spacetime algebra $C\ell(1,3)$ may be represented in terms of 2×2 matrices with quaternion entries which is equivalent to 8×8 matrices with real entries. Algebras contain a much richer structure than their matrix representation. In the case of Clifford algebras, several different geometries can be represented by the same matrix representation. It is therefore often required to work with the algebras themselves rather than with their matrix representations.

The spacetime algebra $C\ell(1,3)$ is not a division algebra. This has been perceived as a weakness of the algebra and some have concluded that because of this the algebra cannot be the mathematical structure that best describes nature. In chapter 10.1 we verify some of the findings of van der Mark and Williamson [10] who have shown that the areas of the algebra where division is undefined correspond exactly to the limiting cases of physical interest, for example on the light cone. The invertibility or non-invertibility of multivectors gives physical insight into conserved quantities and limitations of physical systems. The behaviour of the algebra is in harmony with the behaviour of nature. The breakdown of division in the algebra is therefore not a weakness but is necessary.

The Clifford group preserves the grade of multivectors. Mono-vectors are mapped to mono-vectors *etc.* under the action of a group element. For certain physical scenarios, this group is too restrictive. In chapter 10.1, a group that preserves not the grade but rather the parity (that is the even or oddness of multivectors) is sought. It is shown that the elements of this extended Clifford group are precisely the invertible homogeneous (that is either even or odd) elements of the Clifford algebra. It is therefore important to determine what multivectors are invertible.

In chapter 5, we show that the Lorentz force law, $\mathbf{F}^L = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$, follows from considerations of the total energy of the electromagnetic field. This result challenges the standard claim that the Lorentz force law is a necessary yet separate addition to Maxwell's equations.

We use the usual form for the energy density of the electric and magnetic fields, integrated over all space, to define the total energy U^{EM} of the system. Changes to this total energy when the sources are moved, described by the gradient operator ∇, leads to the Lorentz force law as $\mathbf{F}^L = -\nabla U^{EM}$.

The work presented in this chapter is presently in manuscript form to submit for possible future publication in Foundations of Physics Journal. The concept for this chapter came out of conversations with Martin van der Mark when he visited Canterbury in 2008 and from his paper co-authored by John Williamson [11] in which they derive the origin of the exclusion principle by considering fields acting on fields.

Maxwell's equations can be formulated within the Clifford algebra $C\ell(1,3)$. In this formulation, Maxwell's equations are written as the single geometric equation. Starting with a mono-vector potential A and assuming that the scalar part of dA vanishes (Lorenz condition), gives Maxwell's equations as $d^2A = -dF = J$, where F is a bi-vector and J is a source term. In the first half of chapter 6 we review this derivation of Maxwell's equations.

Weinberg derived the field equations for arbitrary spin particles [12]. Taking the massless limit of the field equations of a spin one particle should give Maxwell's equations. This is however not the case and one obtains a more general set of equations referred to as the *generalised* Maxwell equations, containing two extra scalar fields in addition to the electric and magnetic fields. In the second half of the same chapter we show how this set of equations may be derived in $C\ell(1,3)$ by starting with a more general potential, a mono-vector plus tri-vector, and not assuming any gauge conditions (such as Lorenz condition). As with the ordinary Maxwell equations, these equations may be written as a single geometric equation in $C\ell(1,3)$.

A natural extension of physical laws to the Planck scale can be obtained by a Lie algebraic modification of the Poincaré and Heisenberg algebras in such a way that the resulting algebra is *immune* to infinitesimal perturbations in its structure constants. This resulting algebra is commonly referred to as the stabilised Poincaré-Heisenberg algebra, or SPHA for short. There is a limit in which the SPHA reduces to the conformal algebra. This limit may describe how the present day Poincaré-algebraic description relates to the conformal-algebraic description of the universe in the past.

We establish in chapter 7 that theories of the aforementioned class inevitably lead to changes in either the homogeneity or continuity of the underlying physical space and that any cosmologically derived quantum effects may carry a strong polarisation and spin dependence.

The SPHA leads us to consider two additional length scales ℓ_P and ℓ_C and a dimensionless parameter β. We show that the parameter β is closely related to the geometry of the underlying physical space and if nonzero will radically affect some of the quantum relativistic and cosmological notions.

The work presented in this chapter is based on the publication [5]. The main contribution from the author of this thesis to the cited publication is calculating the conformal algebraic limit and discussing its physical implications.

In chapter 8 it is shown that the non-scalar basis elements of $C\ell(1,3)$ generate the SPHA under the Lie bracket $[x, y] = xy - yx$. This approach avoids the traditional stability considerations.

We show that the dimensionless parameter β induces a mixing of the X_μ and P_μ operators, not only in the conformal algebraic limit but in the SPHA in general. Furthermore, an expression for β may be found in terms of two length scales ℓ_P, ℓ_C and an angle parameter φ. This result may contribute to further understanding how β affects various quantum relativistic and cosmological notions.

The work in chapter 8 is based on the publication [4].

The study of the conformal group in $\mathbb{R}^{p,q}$ usually involves the conformal compactification of $\mathbb{R}^{p,q}$. This allows the transformations to be represented by linear transformations in $\mathbb{R}^{p+1,q+1}$. So, for example, the conformal group of Minkowski space, $\mathbb{R}^{1,3}$ leads to its isomorphism with $SO(2,4)$. This embedding into a higher dimensional space comes at the expense of the geometric properties of the transformations. This is particularly true of $\mathbb{R}^{1,3}$ where we might well prefer to keep the geometric nature of the various types of transformations in sight.

In chapter 9 of this thesis we show that this linearization procedure can be achieved with no loss of geometric insight, if, instead of using this compactification, we let the conformal transformations act on two copies of the associated Clifford algebra. Although we are mostly concerned with the conformal group of Minkowski space (where the geometry is clearest), generalization to the general case is straightforward.

This chapter is based on the manuscript [7] which has been accepted for publication in the Bulletin of the Belgian Mathematical Society.

1.3 List of Publications

- P. H. Butler, A. B. Gillard, N. G. Gresnigt, W. P. Joyce, B. M. S. Martin, P. F. Renaud, *Physics with a real Clifford algebra* Proceedings of the Professor Brian G. Wybourne Commemorative Meeting, 2005.

- N. G. Gresnigt, P. F. Renaud, P. H. Butler, *The Stabilized Poincaré-Heisenberg Algebra: a Clifford Algebra Viewpoint* Int. J. Mod. Phys. D, 2007, 16, 1519-1529

- D. Ahluwalia, N. Gresnigt, A. Nielsen, D. Schritt, T. Watson, *Possible polarization and spin-dependent aspects of quantum gravity* International Journal of Modern Physics D, World Scientific, 2008, 17, 495

- N. G. Gresnigt, P. F. Renaud, *On the Geometry of the Conformal Group in Spacetime* accepted for publication in Bulletin of the Belgian Mathematical Society, 2009.

- P. H. Butler, N. G. Gresnigt, P. F. Renaud, *The Lorentz force from energy considerations* University of Canterbury internal manuscript, 2009

- P. H. Butler, N. G. Gresnigt, P. F. Renaud, *Assumptions at the foundations of physics* University of Canterbury internal manuscript, 2009

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I thank Dharam Vir Ahluwalia for his insight and expertise relating to the stabilised Poincaré-Heisenberg algebra and physics in general. Without his support and input, it is unlikely, chapters 7 and 8 would be part of this thesis.

Gratitude is also due to Dr. Martin van der Mark (Philips Research, Eindhoven) and Dr. John Williamson (University of Glasgow) for their collaboration from the other side of the planet. It was exciting to share our ideas and views when Martin van der Mark visited Canterbury for 10 weeks in 2008. Indeed the concept for chapter 5 arose from discussion (after several beers I am told) between Philip and Martin. The (funny and sometimes lengthy) e-mails from John Williamson have been a source of encouragement to get our ideas organised in a coherent manner and try to get them published.

I also acknowledge fellow students, Jake Gulliksen and Adam Gillard for their insights, thoughts and discussions over the years. I especially acknowledge working together with Jake Gulliksen on some of the issues discussed in chapter 5. Dr. William Joyce and Dr. Benjamin Martin are also acknowledged for their help especially during my honours year and first year of my Ph.D.

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My gratitude is also extended to Philip Butler, Peter Renaud, Samuel M^{c} Cahon and Sebastian Horvath for proofreading this thesis and feedback. Any remaining typos and errors are left as an exercise for the reader to find and correct.

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Chapter 2

Construction of the algebra $C\ell(1,3)$ from geometry

2.1 Introduction

In this chapter of the thesis, a number of arguments are presented to show that the Clifford algebra $C\ell(1,3)$ is the correct mathematical structure for formulating physics in spacetime. This algebra is derived from the underlying geometry of space-time (first considering three space and the Clifford algebras of two and three dimensions) to set the scene for the other chapters of this thesis where the utility of this algebra is demonstrated.

Books by Penrose [1], Smolin [2] and Woit [3] make a plea for a renewed attempt at getting the foundations of physics right. In these texts it is suggested that there need to be changes to at least one but more probably several of the current understandings of special relativity, quantum mechanics, quantum field theory, the standard model of particle physics and general relativity.

Penrose [1] ends his case (page 1045) with

> [T]here are [many] deeply mysterious issues about which we have very little comprehension. It is quite likely that the 21st century will reveal even more wonderful insights than those we have been blessed with in the 20th. But for this to happen, we shall need powerful new ideas, which will take us in directions significantly different from those currently being pursued. Perhaps what we mainly need is some subtle change in perspective—something we have all missed...

Since 1989, Philip Butler and Peter Renaud have sought a Clifford algebra alternative to the Dirac algebra for relativistic quantum mechanics. The twenty years of effort (together with others) have led to some publications on the geometric foundations of physics [13, 14]. This thesis reports on some of the progress that has been made over the last three years. This progress has led to a number of papers and there is currently an effort by us (led by Philip Butler) to produce a set of papers that carefully derives these right mathematical structures and consequently formulates physics from them.

During this time we have been seeking the right mathematical structure to describe the foundations of physics. Such a mathematical structure should be firmly grounded in the underlying geometry of spacetime. Therefore, in this chapter, and to a greater extent in [8], we go back to this underlying geometry and from this derive the mathematical structure that incorporates this geometry. At the foundational level, concepts and ideas continue to come together to gradually form a coherent picture of space, time and energy and there is growing evidence that the appropriate mathematical structure to be used is that of the Clifford algebra $C\ell(1,3)$.

Presented is an overview of why we believe this to be the appropriate structure. We motivate this mathematical structure from the geometry of spacetime. The arguments and calculations presented here are a summary of those found in [9]. In that paper we spell out the primitive 'physical postulates' that we use to think about how the real world operates, and then deduce a set of corresponding 'mathematical axioms' where each axiom is firmly tied to the postulated behavior of the physical world. The reader is urged to read this reference where more space is devoted to constructing various arguments rigorously leaving no stone unturned and no assumptions hidden.

The focus in this chapter is on the derivation of $C\ell(1,3)$ from the geometry of spacetime. The focus is not on discussing in detail Clifford algebras themselves and how one may formulate physical theories using these algebras. The reason for this is twofold. The first is that there exists extensive literature on the formulation of physics in terms of Clifford algebras. The interested reader is referred to the works of Hestenes [15–18], Doran and Lasenby [19] and Gull *et. al.* [20]. Lounesto [21] is also a great text containing a collection of many (although somewhat unstructured) ideas and results. For a good summary of the algebra $C\ell(1,3)$ the reader is directed to the Ph.D. thesis of Leary [22]. The second is that to the author's awareness, there are very few texts that derive the algebras from the underlying geometry of the space and conclude that $C\ell(1,3)$ is the necessary choice for formulating physics in spacetime.

In the next section we provide a brief historical overview of Clifford algebras, wherein the discoveries made by Hamilton and Clifford in the 19th century are discussed. Section (2.3) discussess the geometry of two dimensional space. The geometric product is introduced that gives rise to a four dimensional Clifford algebra which on top of a scalar and two basis vectors, also contains a different mathematical object called a bi-vector. It is shown that this bi-vector represents a plane and is useful for describing rotations in the plane. In section (2.4) we generalise to three spatial dimensions. Rotations in three space do not commmute and so a non-commutative algebra is needed to describe rotations. The eight dimensional Clifford algebra $C\ell(0,3)$ provides the right mathematical objects to describe three space. It is shown that the bi-vectors are isomorphic to the quaternions and both structures are suitable for representing rotations. It is also shown that of the two algebras $C\ell(3,0)$ and $C\ell(0,3)$, only the second of these algebra preserves handedness. Finally, in section (2.5) four dimensional spacetime is considered and it is shown that the appropriate Clifford algebra to describe spacetime is the sixteen dimensional algebra $C\ell(1,3)$.

2.2 A brief history of Clifford algebra and quaternions

We present a brief historical overview of the quaternions and Clifford algebras. The reader is referred to the text by Altmann [23] for a more detailed and complete review of the subject.

In the early years of the 19th century, the nature of negative and imaginary numbers was of interest to mathematicians and physicists. Complex numbers were introduced to solve quadratic equations (e.g. $x^2 = -1$). Complex here has the meaning being joined together (as in 'apartment complex', not as in complicated). A complex number has two parts, called the real and imaginary parts. Argand and others showed that complex numbers can be used to calculate both rotations and translations in 2-dimensions (2D). These techniques are widely used in mathematics, physics and engineering (including software engineering). The Argand diagram is used in high school physics under the name of "phasor diagram", where the two dimensions are usually displacement and time.

W. R. Hamilton was motivated to explore geometry algebraically. His goal was to invent an algebra that would do for rotations in three dimensions what complex numbers do for rotations in two dimensions. In 1843, Hamilton [24], through the analysis of translations and rotation in three dimensional space, was led to a generalisation of the complex number algebra that described rotations in 3-dimensions (3D). In this algebra, i, j, k satisfy

$$i^2 = j^2 = k^2 = ijk = -1.$$

This he called the quaternions as they are a 'complex' of 4 parts - in modern maths language it is a vector of 4 components, also known as a 4-tuple. Hamilton's more extensive work on quaternions [25] appeared in 1853. A quaternion has one real, and three imaginary parts. (That is jargon, it means it has one part that squares to a positive real number, and 3 independent parts each of which square to negative numbers.) Unfortunately for the advancement of the subject, Hamilton thought the three imaginary parts were the same as the components of the Newton's vector of translations. Perhaps because of this erroneous assumption, and although Hamilton is perhaps the most famous mechanics scientist of the 1800s, quaternions have never become a mainstream tool for physicists, even though Hamilton's discovery of the quaternion algebra preceded matrix algebra by 14 years. In fact quaternions had been used by very few until some computer scientists recognised their use for rotation in the 1980s.

Hamilton's quaternion algebra was the first example of a non-commutative algebra. Only a year after Hamilton published his algebra, Grassmann published his work on the exterior algebra [26], another non-commutative algebra. Carrying on from the work on Grassmann, in 1878, J.C. Maxwell's student/protégé W.K. Clifford published his paper entitled 'Applications of Grassmann's extensive algebra' [27]. In this paper, Clifford presented a non-commutative geometric product and showed that the spatial algebra for working with Maxwell's electromagnetic equations was an 8-component algebra, the 4 quaternions, the 3 vector components of Newton and an eighth piece, called a pseudoscalar. It was this paper by Clifford that has led to a class of algebras now called Clifford algebras. We refer to Clifford's 16-component algebra later when we consider spacetime.

In the 1960's Hestenes led a revival of interest in Clifford algebra. Today, mainstream teaching and research in the mathematical sciences use complex numbers, and extensions of Newton's 3-vector algebra by Euler, Gibbs, Heaviside, and others. Euler's formulas for rotations require complex numbers and three axes to be chosen and three angles to be evaluated. As the computer scientists have found, quaternions lead to significant conceptual and computational improvements. It is only in the past 25 years that a significant number of mathematicians, physicists and, we trust, engineers have picked up the ideas of either Hamilton or Clifford. The result is many further significant conceptual and computational improvements, including, significant advances in the understanding of elementary particles.

2.3 The structure of two dimensional space

In this section we explore the geometry of two dimensional space and the associated Clifford algebras of this space, $C\ell(2,0)$ and $C\ell(0,2)$. In particular we consider how two vectors in this space may be multiplied together in a geometrically meaningful way. Rotations in two dimensions are discussed and it is shown that multiplying two vectors together will result in an algebraic object called a bi-vector which is the geometric equivalent of a plane. These bi-vectors are also closely related to rotations.

Our starting point is a two dimensional vector space V with an understanding of how vectors may be added is this space. We avoid any in depth discussion and derivation of the concepts of vectors and vector spaces from geometry. Such information can be found in [8] on which this chapter is based, and also in [14]. Requiring that our product operation describes rotations and matches Pythagoras' theorem, gives rise to a Clifford algebra.

2.3.1 The geometric product in two dimensions

Given two vectors $\mathbf{a}$ and $\mathbf{b}$ in a two dimensional vector space V, we want to define the notion of multiplication of the first vector $\mathbf{a}$ with the second vector $\mathbf{b}$. In particular, we want to multiply these two vectors in a geometric manner. We want the product to be both associative and bi-linear in the two vectors. The conventional Heaviside-Gibbs cross product is of no use here since we only have two dimensions to work with rather than three.

In [9] we show that geometry allows us to choose linearly independent basis vectors $\hat{x}, \hat{y}$, that are geometrically orthogonal in the sense that a $\frac{\pi}{2}$ rotation rotates $\hat{x}$ into $\hat{y}$ and $\hat{y}$ into $-\hat{x}$. By writing our two vectors in terms of such a basis, we can write the product as a term-wise associative expansion. Doing so together with requiring that Pythagoras' theorem holds, gives us a definition of the geometric product and leads to the Clifford algebras of two dimensional space. In subsequent sections we generalise the work to higher dimensions.

Start by writing the two vectors in terms of a geometrically orthonormal basis.

$$\mathbf{a} = a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}}, \qquad \mathbf{b} = b_x\hat{\mathbf{x}} + b_y\hat{\mathbf{y}}, \tag{2.1}$$

where a_x, a_y, b_x, b_y are scalars and $\hat{\mathbf{x}}, \hat{\mathbf{y}}$ are the basis vectors. Consider $\mathbf{ab}$ to be the term-wise associative expansion (the free product in [13]) of the components written in our orthonormal axis system.

$$\begin{aligned}\mathbf{ab} &= (a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}})(b_x\hat{\mathbf{x}} + b_y\hat{\mathbf{y}}),\\ &= a_x b_x\hat{\mathbf{x}}^2 + a_y b_y\hat{\mathbf{y}}^2 + a_x b_y\hat{\mathbf{x}}\hat{\mathbf{y}} + a_y b_x\hat{\mathbf{y}}\hat{\mathbf{x}}, \qquad (2.2)\end{aligned}$$

where we have used the property that the components, a_x, a_y, b_x, b_y, being numbers, commute with the unit vectors. However we do *not* assume that the product of the vectors is commutative.

Next we require that Pythagoras' theorem holds. For the product of the vector $\mathbf{a}$ with itself, we have

$$\begin{aligned}\mathbf{aa} &= \mathbf{a}^2,\\ &= a^2\hat{\mathbf{a}}^2,\\ &= a_x^2\hat{\mathbf{x}}^2 + a_y^2\hat{\mathbf{y}}^2 + a_x a_y(\hat{\mathbf{x}}\hat{\mathbf{y}} + \hat{\mathbf{y}}\hat{\mathbf{x}}). \qquad (2.3)\end{aligned}$$

If this is to satisfy Pythagoras, $a^2 = a_x^2 + a_y^2$, then we must have both

$$\hat{\mathbf{a}}^2 = \hat{\mathbf{x}}^2 = \hat{\mathbf{y}}^2, \qquad (2.4)$$

and also

$$\hat{\mathbf{x}}\hat{\mathbf{y}} + \hat{\mathbf{y}}\hat{\mathbf{x}} = 0. \qquad (2.5)$$

Let $\hat{\mathbf{a}}^2$ be the rational number η. After rescaling we have two independent cases for η, depending on whether $\hat{\mathbf{a}}^2$ is positive ($\eta = +1$) or negative ($\eta = -1$). The number η is known as the metric of the space. The choice of $\eta = -1$ gives $\mathbf{a}^2 \leq 0$ for all $\mathbf{a}$. We shall call this choice the 'anti-Euclidean metric'. Later in this chapter we compare and contrast the two possible choices of metric $\eta = \pm 1$.

The pair of equations, eq(2.4) and eq(2.5), define the Clifford algebras $C\ell(2,0)$ and $C\ell(0,2)$ for $\eta = +1$ and $\eta = -1$ respectively.

The second equality, eq(2.5), introduces a fourth element (beyond 1, $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$)

$$\hat{\mathbf{k}} = \hat{\mathbf{x}}\hat{\mathbf{y}} = -\hat{\mathbf{y}}\hat{\mathbf{x}} \qquad (2.6)$$

into the algebra. We have created an associative algebra of the four elements 1, $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{k}}$ where

$$\begin{aligned}\hat{\mathbf{k}}^2 &= (\hat{\mathbf{x}}\hat{\mathbf{y}})(\hat{\mathbf{x}}\hat{\mathbf{y}}),\\ &= -\hat{\mathbf{x}}(\hat{\mathbf{y}}\hat{\mathbf{y}})\hat{\mathbf{x}},\\ &= -\eta\hat{\mathbf{x}}\hat{\mathbf{x}},\\ &= -\eta^2,\\ &= -1, \qquad (2.7)\end{aligned}$$

for both choices of η.

$\hat{\mathbf{k}}$ is the algebraic unit that describes a unit area in the xy-plane. It is not the normal to the plane, such a normal does not exist in our two dimensional geometry. Instead, just as the basis vector $\hat{\mathbf{x}}$ is the direction of the x-axis and is dimensionless, so the basis bi-vector $\hat{\mathbf{k}}$ is what we may call the 'direction' of the xy-plane. Being the product of two dimensionless quantities, $\hat{\mathbf{k}}$ is dimensionless and, as we shall see, is associated with the angle $\pi/2$ radians.

The four objects 1, $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{k}}$ and their negatives form the eight element group, associated with the Clifford algebras $C\ell(2,0)$ or $C\ell(0,2)$ as $\eta = \pm 1$. The group combination law is the associative product defined above. We refer to this product as the *geometric product*. A vector space together with this product gives rise to a Clifford algebra. The geometric product was first introduced by Clifford in 1878 [27]. The four elements 1, $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{k}}$ are the basis for the four dimensional vector space over our field, $\mathbb{R}$ using the addition operation, with the arbitrary element written

$$\mathbf{A} = a + b\hat{\mathbf{x}} + c\hat{\mathbf{y}} + d\hat{\mathbf{k}} \quad \text{where } a, b, c, d \in \mathbb{R}. \tag{2.8}$$

Geometrically, the product $\mathbf{ab}$ of two vectors represents both the angle between the two vectors, and also a segment of the plane spanned by the vectors. Notice however that the information about the individual lengths of $\mathbf{a}$ and $\mathbf{b}$ has been lost in the process of multiplying the two vectors together. This can be seen explicitly using the bi-linearity of the geometric product

$$\mathbf{ab} = (\frac{1}{r}\mathbf{a})(r\mathbf{b}). \tag{2.9}$$

The algebra is that structure that has the operations of the addition of elements, $\mathbf{A} + \mathbf{B}$, the product of scalars a and elements $\mathbf{A}$ as $a\mathbf{A}$, and the product of elements with elements, $\mathbf{AB}$.

We refer to the scalar part of $\mathbf{A}$ as grade zero and the vector part as grade one. This generalises trivially to higher dimensions, for example the bi-vector element $\hat{\mathbf{k}}$ is grade two. We refer to a linear combination of different grade objects as a mixed grade multivector.

2.3.2 Rotations in two dimensions

In the previous subsection, a geometric product of two vectors that preserved Pythagoras' theorem was found. By introducing such a product, our two dimensional vector space gives rise to a four dimensional Clifford algebra consisting of a scalar 1, two vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ and one bi-vector $\hat{\mathbf{k}} = \hat{\mathbf{x}}\hat{\mathbf{y}}$. The focus of this subsection is on describing rotations in

2-dimensional geometry. For a more thorough treatment of rotations in 2-dimensions the reader is directed to [9] and also the works by Gull *et. al.* [20] and Leary [22].

The usual way rotations are handled is about a rotation axis. This method causes difficulties particularly in two dimensions (and four dimensions too) since if we want to rotate the vector $\hat{\mathbf{x}}$ into $\hat{\mathbf{y}}$ for example, the rotation axis needs to be normal to both these vectors. This is however not possible in a two dimensional space. A third dimension would be required. In the early years of the 19th century Argand and others recognized that the complex number algebra could provide a tool for handling rotations in the plane. By means of multiplication by a complex number, vectors can be rotated within the plane. It will be shown that the bi-vector of the previous subsection allows us to describe rotations in the two dimensional plane without the need of a normal vector or an imaginary $i = \sqrt{-1}$.

Consider the bi-vector $\hat{\mathbf{k}}$ which was obtained in the previous subsection from taking the geometric product of the two basis vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$. Notice what happens when $\hat{\mathbf{k}}$ operates on the basis vectors from the left

$$\begin{aligned}
\hat{\mathbf{k}}\hat{\mathbf{x}} &= (\hat{\mathbf{x}}\hat{\mathbf{y}})\hat{\mathbf{x}} = \hat{\mathbf{x}}(\hat{\mathbf{y}}\hat{\mathbf{x}}), \\
&= -\hat{\mathbf{x}}(\hat{\mathbf{x}}\hat{\mathbf{y}}) = -(\hat{\mathbf{x}}\hat{\mathbf{x}})\hat{\mathbf{y}}, \\
&= -\eta\hat{\mathbf{y}}, \\
&= \hat{\mathbf{y}},
\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}
\hat{\mathbf{k}}\hat{\mathbf{y}} &= (\hat{\mathbf{x}}\hat{\mathbf{y}})\hat{\mathbf{y}}, \\
&= \hat{\mathbf{x}}(\hat{\mathbf{y}}\hat{\mathbf{y}}), \\
&= \eta\hat{\mathbf{x}}, \\
&= -\hat{\mathbf{x}}.
\end{aligned} \tag{2.11}$$

We have chosen $\eta = -1$ in the above calculations for it is the anti-Euclidean metric that best describes the geometry of the universe we live in. This will be discussed in greater detail later in this chapter. Via left multiplication, the bi-vector $\hat{\mathbf{k}}$ induces a rotation of $\frac{\pi}{2}$ radians on vectors in the plane spanned by the basis vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$. The order of multiplication is important here. Multiplying on the right by $\hat{\mathbf{k}}$ induces a rotation of $\frac{-\pi}{2}$ on vectors in this plane.

Since $\hat{\mathbf{k}}^2 = -1$, DeMoivres' theorem holds. The result will be familiar to most readers in terms of the complex numbers $i = \sqrt{-1}$. For any object such as $\hat{\mathbf{k}}$ that squares to -1 we have

$$\exp(\hat{\mathbf{k}}\phi) = \mathrm{e}^{\hat{\mathbf{k}}\phi} = \cos\phi + \hat{\mathbf{k}}\sin\phi. \tag{2.12}$$

We may transform the expression for $\mathbf{a}$ in orthonormal (Galilean) coordinates ($\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$) into circular polar coordinates (r, θ), where $r = a$

$$\begin{aligned} \mathbf{a} &= a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}}, \\ a_x &= a\cos\theta, \\ a_y &= a\sin\theta, \\ \mathbf{a} &= a\hat{\mathbf{x}}\cos\theta + a\hat{\mathbf{y}}\sin\theta, \end{aligned} \tag{2.13}$$

and so using eq(2.12) and $\hat{\mathbf{x}}\hat{\mathbf{k}} = -\hat{\mathbf{y}}$

$$\begin{aligned} \mathbf{a} &= a\hat{\mathbf{x}}\exp(-\hat{\mathbf{k}}\theta), \\ &= a\hat{\mathbf{x}}\mathrm{e}^{-\hat{\mathbf{k}}\theta}. \end{aligned} \tag{2.14}$$

The operator $\exp(\hat{\mathbf{k}}\phi)$ of eq(2.12) rotates $\mathbf{a}$, when operating on the left, by angle ϕ

$$\begin{aligned} \mathrm{e}^{\hat{\mathbf{k}}\phi}\mathbf{a} &= \mathrm{e}^{\hat{\mathbf{k}}\phi}a\hat{\mathbf{x}}\mathrm{e}^{-\hat{\mathbf{k}}\theta}, \\ &= a\hat{\mathbf{x}}\mathrm{e}^{-\hat{\mathbf{k}}\phi}\mathrm{e}^{-\hat{\mathbf{k}}\theta}, \\ &= a\hat{\mathbf{x}}\mathrm{e}^{-\hat{\mathbf{k}}(\phi+\theta)}. \end{aligned} \tag{2.15}$$

where we have used the fact that $\hat{\mathbf{x}}$ anti-commutes with $\hat{\mathbf{k}}$. Again, the fact that $\exp(\hat{\mathbf{k}}\phi)$ acts on the left is important here.

The above arguments show that $\exp(\hat{\mathbf{k}}\phi) = \mathrm{e}^{\hat{\mathbf{k}}\phi}$ is the operator, that when acting on the left of a vector, rotates that vector by the angle ϕ in the xy-plane, $\hat{\mathbf{k}}$. However it does not behave this way acting on scalars, or on itself. Thus to write a formula for multi-vectors (scalars, vectors and bi-vectors), we require a different form. This form is as a two–sided operation: If $\mathbf{A}$ is an arbitrary element of $C\ell(0,2)$, as in eq(2.8), then

$$\mathrm{Rot}(\text{by } \phi \text{ in the } \hat{\mathbf{k}} \text{ plane})(\mathbf{A}) = \mathrm{e}^{\hat{\mathbf{k}}\phi/2}\mathbf{A}\mathrm{e}^{-\hat{\mathbf{k}}\phi/2}, \tag{2.16}$$

because $\hat{\mathbf{k}}$ commutes with scalars and itself, and anti-commutes with the mono-vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$. As we have seen, scalars (or numbers) and the bi-vector $\hat{\mathbf{k}}$ are unchanged by rotation in the xy-plane, so $a + d\hat{\mathbf{k}} \Rightarrow a + d\hat{\mathbf{k}}$ while the vector part of $\mathbf{A}$, $b\hat{\mathbf{x}} + c\hat{\mathbf{y}}$ is rotated appropriately

$$\begin{aligned} \mathrm{e}^{\hat{\mathbf{k}}\phi/2}\mathbf{A}\mathrm{e}^{-\hat{\mathbf{k}}\phi/2} &= \mathrm{e}^{\hat{\mathbf{k}}\phi/2}(a + b\hat{\mathbf{x}} + c\hat{\mathbf{y}} + d\hat{\mathbf{k}})\mathrm{e}^{-\hat{\mathbf{k}}\phi/2}, \\ &= (a + d\hat{\mathbf{k}})\mathrm{e}^{\hat{\mathbf{k}}\phi/2}\mathrm{e}^{-\hat{\mathbf{k}}\phi/2} + (b\hat{\mathbf{x}} + c\hat{\mathbf{y}})\mathrm{e}^{-\hat{\mathbf{k}}\phi/2}\mathrm{e}^{-\hat{\mathbf{k}}\phi/2}, \\ &= (a + d\hat{\mathbf{k}}) + (b\hat{\mathbf{x}} + c\hat{\mathbf{y}})\mathrm{e}^{-\hat{\mathbf{k}}\phi}. \end{aligned} \tag{2.17}$$

Reversing the order of multiplication results in a relative minus sign in the exponential.

The general bi-vector **ab**, eq(2.2), can be written in terms of the scalar and bi-vector eq(2.15) to give

$$\begin{aligned}
\mathbf{a} &= a\hat{\mathbf{x}}\mathrm{e}^{-\hat{\mathbf{k}}\theta_1}, \\
\mathbf{b} &= b\hat{\mathbf{x}}\mathrm{e}^{-\hat{\mathbf{k}}\theta_2}, \\
\mathbf{ab} &= ab\hat{\mathbf{x}}\mathrm{e}^{-\hat{\mathbf{k}}(\theta_1)}\hat{\mathbf{x}}\mathrm{e}^{-\hat{\mathbf{k}}\theta_2}, \\
&= ab\mathrm{e}^{\hat{\mathbf{k}}(\theta_1-\theta_2)}.
\end{aligned} \tag{2.18}$$

Reversing the order of multiplication results in a relative minus sign in the exponential.

Bi-vectors are, in an active sense, operators that rotate all elements of the Clifford algebra $C\ell(0,2)$. Scalars and pure bi-vector elements are unchanged under rotation in the plane, while for vectors rotation has a simple formula. By inspection of equation (2.18), if **a** and **b** are of the same length, then

$$\mathrm{Rot}(\mathbf{a}\rightarrow\mathbf{b})(\mathbf{r}) = \mathbf{rab}/\mathbf{a}^2. \tag{2.19}$$

2.4 The structure of three dimensional space

In the previous section we discussed two dimensional space. Defining a geometric product on this space led to a four dimensional real algebra to be defined on the plane. This algebra contained a bi-vector element which was shown to be useful in describing rotations in the plane. In this section the number of dimensions is increased to three. Three dimensional space is usually described in terms of three orthogonal unit vectors. On top of three unit vectors $\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}}$, three dimensional space also contains three planes $\hat{\mathbf{i}},\hat{\mathbf{j}},\hat{\mathbf{k}}$ normal to the unit vectors. Thus one would need at least a six dimensional algebra. Usually a complex space is used to provide the required six dimensions. It will be shown here that a much better treatment of three space is possible using an eight dimensional real Clifford algebra derived from the geometry.

There is widespread confusion in the literature between the (polar) vectors $\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}}$ and the planes (axial vectors) $\hat{\mathbf{i}},\hat{\mathbf{j}},\hat{\mathbf{k}}$ that arise as the product of two vectors. The two sets are often used interchangeably to denote the basis vectors. That is, the planes $\hat{\mathbf{i}},\hat{\mathbf{j}},\hat{\mathbf{k}}$ are often identified with the vectors $\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}}$. Such an identification obscures the geometric distinction between them. This confusion originated with Hamilton who in his initial work was at pains to distinguish his versors ('rotors' or 'axial vectors') from his lines ('translations' or 'polar vectors'). Unfortunately he mixed the two up, and the mess between the lines (or unit vectors) $\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}}$ and the planes (or unit rotators) $\hat{\mathbf{i}},\hat{\mathbf{j}},\hat{\mathbf{k}}$ continues to this day. Many physics text books use the notation $\hat{\mathbf{i}},\hat{\mathbf{j}},\hat{\mathbf{k}}$ for the unit vectors. Simon Altmann [23] gives the fullest history of this mess that we are aware of. We explore the confusion between the unit vectors of 3-space $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ and the 'vectors' arising as the product of pairs, as in the cross product $\hat{\mathbf{x}} \times \hat{\mathbf{y}}$. These objects may better be called bi-vectors and may be distinguished from vectors by their properties under reflection.

Hamilton [24, 25] spent many years seeking a generalization of the algebra of complex numbers that seemed, via the Argand diagram, to give a good mathematical description of the geometry of the plane. Complex numbers gave the mathematics describing translations in two dimensions, as addition of pairs of numbers for coordinates in the x and y directions. Complex numbers describe rotation by multiplications. However the structure of 3-complexes that he sought does not exist, but he did find the necessary 4-dimensional generalization, which he called the quaternions whose basis elements satisfy

$$1^2 = 1, \quad i^2 = j^2 = k^2 = ijk = -1. \tag{2.20}$$

The story of Hamilton recognizing what was needed is part of the oft quoted folklore of mathematical discovery. It came to him "in a flash" while walking with his wife along a Dublin canal on a Sunday afternoon. The generalization for three dimensions of the "ordered pair" or 2-complex needed for two dimensional geometry, was to an "ordered 4-tuple" or "quaternion". Furthermore the quaternion components needed to form a non-commutative algebra. Hamilton's quaternion algebra was the first formal non-commutative algebra, as it must be to represent the non-commutativity of rotations in three dimensions.

Hamilton was en route to developing the appropriate algebra for describing the geometry of space. In his lectures to the Dublin Royal Society in 1853 [25] he carefully distinguished between polar vectors (which he called lines) and axial vectors (which he called versors). Polar vectors describe the positions of the points of objects, and also describe translations. Axial vectors describe the orientations of (non-point) objects and also describe rotations. He then proceeded (page 71) to write *both* polar vectors and axial vectors in his axial basis which he labeled by the letters i, j, k. Unfortunately for the development of the subject he fails to maintain this distinction, and writes

> And I conceive that we may *now* legitimately, and with advantage, avail ourselves of the same analogy, or of the theorem to which it corresponds, to *dispense* with that *symbolic distinction* which has been above observed, between the three quadrantal *versors i, j, k*, and the three *lines* i, j, k, which have respectively the directions of their three axes. [Italics as in the original.]

We presume he made this identification so as to keep his algebra small. Had he retained the distinction he would probably been led to the conclusions of Clifford [27].

The appropriate algebra has the three lines i, j, k, which are now called the basis polar vectors and we write as $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$. It also has the three versors *i, j, k*, which are now called the basis axial vectors and we write as $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$. The complete algebra closes with the addition of two more basis elements, the scalar, 1, and an element we write as $\hat{\mathbf{v}}$ as we shall see it relates to a basis volume element. Thus to describe three dimensional geometry we are best to use the eight dimensional algebra we label $C\ell(0,3)$. One important subalgebra of $C\ell(0,3)$ is the four dimensional quaternion algebra $\mathbb{H}$.

Hamilton proved that the axial vectors i, j, k square to -1, and that they form his famous quaternion algebra

$$i^2 = j^2 = k^2 = ijk = -1.$$

However the incorrect identification between the versors i, j, k and the lines i, j, k continues in the labeling, by many physics texts, of the polar vector basis elements as $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$ where $\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$. This product is correct for axial vectors but not for polar vectors. This unfortunate identification by Hamilton has to be patched up by ignoring the distinction of polar and axial vectors, or equivalently by identifying lines with planes (or translations with rotations). Put yet another way, planes are identified with the line that is normal to the plane.

A further consequence of the polar-axial identification is that the vector algebra is too small to describe the geometry and the physics contained in that geometry. This evidences itself in the need in the usual three dimensional algebra to use complex numbers, effectively a six dimensional space, to describe rotations. The eight dimensional Clifford algebra contains all we need without complex numbers.

2.4.1 Rotations in three dimensions

In this section we explore rotations in three dimensional space. Whereas in the previous section on two dimensional geometry there was just one plane of rotation, there are now three linearly independent planes of rotation normal to three orthogonal basis vectors. It is shown that the rotational ideas of the previous section easily generalise to three dimensions. The presence of an additional basis vector leads us from a four dimensional algebra to an eight dimensional one consisting of: a scalar 1, three basis vectors $\hat{\mathbf{x}}$,$\hat{\mathbf{y}}$,$\hat{\mathbf{z}}$, three bi-vectors $\hat{\mathbf{i}} = \hat{\mathbf{y}}\hat{\mathbf{z}}$, $\hat{\mathbf{j}} = \hat{\mathbf{z}}\hat{\mathbf{x}}$, $\hat{\mathbf{k}} = \hat{\mathbf{x}}\hat{\mathbf{y}}$, and one tri-vector (pseudoscalar) $\hat{\mathbf{v}} = \hat{\mathbf{x}}\hat{\mathbf{y}}\hat{\mathbf{z}}$. The three basis bi-vectors describe the rotations in three space. They do not commute, but give rise to the quaternion algebra.

We showed in the previous section that by demanding that the product of two vectors in the plane satisfies Pythagoras' theorem that

$$\begin{aligned}\hat{\mathbf{x}}^2 &= \hat{\mathbf{y}}^2 = \eta, \\ \hat{\mathbf{x}}\hat{\mathbf{y}} &= -\hat{\mathbf{x}}\hat{\mathbf{y}} = \hat{\mathbf{k}}.\end{aligned}$$

Consider now the term-wise associate expansion of the product of two vectors $\mathbf{a} = a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}} + a_z\hat{\mathbf{z}}$ and $\mathbf{b} = b_x\hat{\mathbf{x}} + b_y\hat{\mathbf{y}} + b_z\hat{\mathbf{z}}$ in three dimensional space.

By demanding again that Pythagoras' theorem holds for such a multiplication, we obtain

$$\hat{\mathbf{x}}^2 = \hat{\mathbf{y}}^2 = \hat{\mathbf{z}}^2 = \eta, \tag{2.21}$$

and

$$\hat{\mathbf{x}}\hat{\mathbf{y}} = -\hat{\mathbf{y}}\hat{\mathbf{x}} = \hat{\mathbf{k}}, \tag{2.22}$$

$$\hat{\mathbf{y}}\hat{\mathbf{z}} = -\hat{\mathbf{z}}\hat{\mathbf{y}} = \hat{\mathbf{i}}, \tag{2.23}$$

$$\hat{\mathbf{z}}\hat{\mathbf{x}} = -\hat{\mathbf{x}}\hat{\mathbf{z}} = \hat{\mathbf{j}}, \tag{2.24}$$

where $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ are defined as shown. Furthermore, the product of the three basis vectors gives the tri-vector which is the pseudoscalar $\hat{\mathbf{v}} = \hat{\mathbf{x}}\hat{\mathbf{y}}\hat{\mathbf{z}}$.

The definitions are chosen retain the cyclic order of the basis vectors ($\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$) when defining the basis bi-vectors ($\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$), and the basis tri-vector $\hat{\mathbf{v}}$. The eight elements $\{1, \hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}, \hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}, \hat{\mathbf{v}}\}$ form the basis of the Clifford algebra $C\ell(0,3)$ when $\eta = -1$ and the Clifford algebra $C\ell(3,0)$ when $\eta = +1$.

The above results for ($\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$) and definitions for ($\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$) lead to the properties

$$\hat{\mathbf{i}}^2 = (\hat{\mathbf{y}}\hat{\mathbf{z}})(-\hat{\mathbf{z}}\hat{\mathbf{y}}) = -\eta^2 = -1$$

$$\hat{\mathbf{i}}\hat{\mathbf{j}} = (\hat{\mathbf{y}}\hat{\mathbf{z}})(\hat{\mathbf{z}}\hat{\mathbf{x}}) = -\eta\hat{\mathbf{k}} \tag{2.25}$$

$$= \hat{\mathbf{k}} \text{ if and only if } \eta = -1 \tag{2.26}$$

$$\hat{\mathbf{v}}^2 = -\eta \tag{2.27}$$

Thus if the anti-Euclidean metric, $\eta = -1$ is chosen, the basis bi-vectors satisfy

$$\hat{\mathbf{i}}^2 = \hat{\mathbf{j}}^2 = \hat{\mathbf{k}}^2 = -1, \tag{2.28}$$

$$\hat{\mathbf{i}}\hat{\mathbf{j}} = -\hat{\mathbf{j}}\hat{\mathbf{i}} = \hat{\mathbf{k}}, \tag{2.29}$$

$$\hat{\mathbf{j}}\hat{\mathbf{k}} = -\hat{\mathbf{k}}\hat{\mathbf{j}} = \hat{\mathbf{i}}, \tag{2.30}$$

$$\hat{\mathbf{k}}\hat{\mathbf{i}} = -\hat{\mathbf{i}}\hat{\mathbf{k}} = \hat{\mathbf{j}}. \tag{2.31}$$

These equations are the relations that characterize Hamilton's [24] quaternions

$$i^2 = j^2 = k^2 = ijk = -1 \tag{2.32}$$

as the relations of eqs(2.28)-(2.31) can be readily derived from eqs(2.32).

2.4.2 The geometric product in three dimensions

The product of two vectors in three space contains both scalar terms and bi-vector terms. If the order of multiplication was reversed, the scalar term would remain the same whereas there would be a sign change for the bi-vector terms. In general, the geometric product $\mathbf{ab}$ of two vectors $\mathbf{a}$ and $\mathbf{b}$ contains a symmetric and antisymmetric part. We write the geometric product as

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}, \tag{2.33}$$

where

$$\mathbf{a} \cdot \mathbf{b} = \frac{1}{2}(\mathbf{ab} + \mathbf{ba}), \tag{2.34}$$

$$\mathbf{a} \wedge \mathbf{b} = \frac{1}{2}(\mathbf{ab} - \mathbf{ba}), \tag{2.35}$$

are the symmetric (dot) and antisymmetric (wedge) parts of the geometric product respectively. We see that the product combines into one definition both the scalar and axial (bi-vector) terms. In the case where two vectors are multiplied together, the symmetric part of the product is the familiar inner product or dot product of vector algebra. The antisymmetric part is similar to the cross product as will now be shown.

Consider the Heaviside-Gibbs cross product $\mathbf{a} \times \mathbf{b}$ of two vectors $\mathbf{a}$ and $\mathbf{b}$. The product yields a third vector orthogonal to both $\mathbf{a}$ and $\mathbf{b}$. That is, it yields a vector that is normal to the plane in which the two vectors lie. The wedge product $\mathbf{a} \wedge \mathbf{b}$ of the two vectors does not produce a vector that is normal to the plane in which $\mathbf{a}$ and $\mathbf{b}$ lie, rather it produces the plane in which the vectors lie. Explicitly we have

$$\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}, \quad \hat{\mathbf{x}} \wedge \hat{\mathbf{y}} = \hat{\mathbf{x}}\hat{\mathbf{y}} = \hat{\mathbf{k}}, \tag{2.36}$$

$$\hat{\mathbf{y}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}}, \quad \hat{\mathbf{y}} \wedge \hat{\mathbf{z}} = \hat{\mathbf{y}}\hat{\mathbf{z}} = \hat{\mathbf{i}}, \tag{2.37}$$

$$\hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}}, \quad \hat{\mathbf{z}} \wedge \hat{\mathbf{x}} = \hat{\mathbf{y}}\hat{\mathbf{z}} = \hat{\mathbf{j}}. \tag{2.38}$$

We say that the wedge product is the Hodge dual of the cross product, where the Hodge dual is defined by right multiplication by the pseudoscalar, in this case the tri-vector $\hat{\mathbf{v}}$[1]. It is readily seen that the mono-vectors $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ are dual to the bi-vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{z}}$

$$\hat{\mathbf{x}}\hat{\mathbf{v}} = -\hat{\mathbf{i}}, \quad \hat{\mathbf{y}}\hat{\mathbf{v}} = -\hat{\mathbf{j}}, \quad \hat{\mathbf{z}}\hat{\mathbf{v}} = -\hat{\mathbf{k}}. \tag{2.39}$$

[1]For $C\ell(0,3)$, the Hodge dual may also be defined by left multiplication because the pseudoscalar is central in this algebra. For any algebra where the pseudoscalar is not central, such as $C\ell(0,2)$ (or $C\ell(1,3)$ discussed in the next section) the Hodge dual must be defined by right multiplication do avoid relative minus signs.

Thus for example, taking the $\hat{\mathbf{i}}$ component of $\mathbf{a} \wedge \mathbf{b}$ gives the $\hat{\mathbf{x}}$ component of $\mathbf{a} \times \mathbf{b}$

$$(\mathbf{a} \wedge \mathbf{b})|_i = (\mathbf{a} \times \mathbf{b})|_x = (a_y b_z - a_z b_y). \tag{2.40}$$

The key result of this subsection is that $\mathbf{a} \wedge \mathbf{b}$ is a pure bi-vector that represents the plane in which $\mathbf{a}$ and $\mathbf{b}$ lie, whereas $\mathbf{a} \times \mathbf{b}$ is a vector that represents lines normal to that plane.

One advantage the wedge product has over the cross product is that it is defined for any dimension. The cross product on the other hand only works in three dimensional space. It requires that there exist exactly one vector perpendicular to the two vectors you are taking the cross product of. In two dimensions no such perpendicular vector exists and in four dimensions, there is no unique perpendicular vector but rather an infinite number of them.

2.4.3 Handedness and the choice of metric

In this subsection we discuss what metric best describes three dimensional space, the Euclidean metric with signature $(+, +, +,)$ or the anti-Euclidean metric with signature $(-, -, -)$. From our analysis it is deduced that the three space Clifford algebra $C\ell(0, 3)$ with the anti-Euclidean metric gives a better mathematical description of physics than the Clifford algebra $C\ell(3, 0)$ with the Euclidean metric. A proper analysis of both translations and particularly rotations in three dimensional space leads naturally to the anti-Euclidean metric. When a Euclidean metric is used, complex numbers need to be introduced to patch up the handedness issues that arise when combining rotations.

Consider a right handed coordinate system $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ together with three basis rotation operators i, j, k, each being a rotation in the yz,zx,xy planes respectively. By using the cross product, the axial vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ usually end up being represented by the polar vectors $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$. However, $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ are directed bi-vectors which are obtained from the wedge product of two vectors. There is thus a distinction between the vectors $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ and the bi-vector $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ that is usually omitted or confused. Keeping the distinction clear between axial vectors (bi-vectors) and polar vector (vectors), one is led to the Clifford algebra $C\ell(0, 3)$ with the anti-Euclidean metric as the only consistent description of the rotational structure of three dimensional space. This is because the $(-, -, -)$ metric preserves the handedness for all products in the associated Clifford algebra whereas the $(+, +, +)$ metric preserves handedness only for polar vectors (vectors) but mixes handedness for axial vector (bi-vector) products. We conclude that therefore an anti-Euclidean metric is required do describe both translations and rotations correctly[2].

[2]Acknowledgment is due to Martin van der Mark who introduced the author to the arguments pre-

Consider a three dimensional set of basis vectors $\{e_1, e_2, e_3\}$ of arbitrary handedness (that is left or right handed) according to some system of reference $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$ which has either a Euclidean $(+,+,+)$ or anti-Euclidean $(-,-,-)$ metric. Make the identifications

$$e_1 \to \hat{\mathbf{x}},$$
$$e_2 \to \hat{\mathbf{y}},$$
$$e_3 \to \hat{\mathbf{z}}.$$

Next, consider the Clifford product between the basis vectors (the symmetric part is zero).

$$e_1e_2 = e_{12}, \qquad e_2e_3 = e_{23}, \qquad e_3e_1 = e_{31}.$$

These products themselves form a new set of basis vectors, the bi-vectors $\{e_{12}, e_{23}, e_{31}\}$, which for consistency must have the same handedness as our original set of basis vectors $\{e_1, e_2, e_3\}$. The notation $e_ie_j = e_{ij}$ has been adopted for simplicity. Define the projection

$$e_{12} \to \hat{\mathbf{k}},$$
$$e_{23} \to \hat{\mathbf{i}},$$
$$e_{31} \to \hat{\mathbf{j}}.$$

Now let us consider the products of these new basis vectors among themselves.

$$\begin{aligned} e_{12}e_{23} &= -e_2^2e_{31} \to \hat{\mathbf{k}}\hat{\mathbf{i}} = -\hat{\mathbf{y}}^2\hat{\mathbf{j}}, \\ e_{23}e_{31} &= -e_3^2e_{12} \to \hat{\mathbf{i}}\hat{\mathbf{j}} = -\hat{\mathbf{z}}^2\hat{\mathbf{i}}, \\ e_{31}e_{12} &= -e_1^2e_{23} \to \hat{\mathbf{j}}\hat{\mathbf{k}} = -\hat{\mathbf{x}}^2\hat{\mathbf{k}}. \end{aligned}$$

Consistency again requires that the left hand sides of the above equations are equal to $\hat{\mathbf{j}}, \hat{\mathbf{i}}$ and $\hat{\mathbf{k}}$ respectively. This in turn implies

$$e_1^2 = e_2^2 = e_3^2 = -1 \to \hat{\mathbf{x}}^2 = \hat{\mathbf{y}}^2 = \hat{\mathbf{z}}^2 = -1.$$

Independent of the handedness of the initial reference system, only the anti-Euclidean metric $(-,-,-)$ gives a consistent covering. Consideration of the handedness of the product of mixed grade objects results in the same conclusion.

The handedness of rotations is conserved within the algebra $C\ell(0,3)$. Two right handed bi-vectors combine via the geometric product to give a third right handed bi-vector.

$$e_{12}e_{23} = e_{31} \text{ in } C\ell(0,3). \tag{2.41}$$

sented here and Stephen Leary who has included it in his thesis [22]. The work in this subsection complements the work done earlier by Butler and McAven [13].

This is however not the case in the alternative algebra $C\ell(3,0)$ where two right handed bi-vectors do not combine to give another right handed bi-vector! Instead we obtain

$$e_{12}e_{23} = -e_{31} \text{ in } C\ell(3,0). \tag{2.42}$$

This means that in this algebra, the bi-vectors cannot be identified with the quaternions, or any consistent set of right handed rotors for that matter.

The anti-Euclidean metric describes the symmetry properties of the physical space that we seem to belong to, that of a space that preserves parity. By that we mean that the parity operation is not a physical operation, but it is a mathematical operation in that we can consider how such an operation is described by the mathematics. One aspect of parity conservation is the preservation of the cyclic ordering of the unit polar vectors $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$, the corresponding axial vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$, and the relations between them, such as $\hat{\mathbf{x}}\hat{\mathbf{y}} = \hat{\mathbf{k}}$.

2.5 Spacetime and the algebra $C\ell(1,3)$

In the previous sections we have discussed translations and rotations in two and three dimensional space. We have so far not yet introduced time. In this section we introduce time as a fourth dimension. Because time has many similarities (and also some important differences) to position, it seems reasonable to treat time as a fourth coordinate. Spacetime can then be treated as a four dimensional vector space. We will show that the geometry of spacetime gives rise to the Clifford algebra $C\ell(1,3)$.

We begin by adding to the three orthonormal spatial unit vectors $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})$ the time unit vector $\hat{\mathbf{t}}$. The speed of light, or indeed the speed of a moving body, is defined as the ratio of distance traveled to the travel time, $v = \ell/t$. If light is emitted at the event $(t_1\hat{\mathbf{t}}+x_1\hat{\mathbf{x}}+y_1\hat{\mathbf{y}}+z_1\hat{\mathbf{z}})$ and received at the event $(t_2\hat{\mathbf{t}}+x_2\hat{\mathbf{x}}+y_2\hat{\mathbf{y}}+z_2\hat{\mathbf{z}})$, then the constancy of the speed of light implies that

$$c(t_1 - t_2) = \pm\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}, \tag{2.43}$$

or

$$c^2(t_1 - t_2)^2 - (x_1 - x_2)^2 - (y_1 - y_2)^2 - (z_1 - z_2)^2 = 0. \tag{2.44}$$

This is the Lorentz invariant metric required. It is the generalisation of the Pythagoras result for orthonormal axes $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$. Pythagoras' theorem says that the length (squared) of a line between two points in 3-space is the sum of the components, and is invariant. The Lorentz metric adds to that by including a fourth time axis which has opposite metric sign to the three space axes. The result of this is that the invariant length (squared) of a line between two points in spacetime can be negative.

Repeating the argument from section 2.3.1, we want the free product $(c(t_1 - t_2)\hat{\mathbf{t}} + (x_1 - x_2)\hat{\mathbf{x}} + (y_1 - y_2)\hat{\mathbf{y}} + (z_1 - z_2)\hat{\mathbf{z}})^2$ to reproduce eq(2.44). We deduce that we need to add to the product rules for the spatial unit vectors, equations (2.21)-(2.24), the additional rules

$$
\begin{aligned}
\hat{\mathbf{t}}^2 &= +1, && (2.45)\\
\hat{\mathbf{t}}\hat{\mathbf{x}} &= -\hat{\mathbf{x}}\hat{\mathbf{t}}, && (2.46)\\
\hat{\mathbf{t}}\hat{\mathbf{y}} &= -\hat{\mathbf{y}}\hat{\mathbf{t}}, && (2.47)\\
\hat{\mathbf{t}}\hat{\mathbf{z}} &= -\hat{\mathbf{z}}\hat{\mathbf{t}}. && (2.48)
\end{aligned}
$$

These rules define the Clifford algebra $C\ell(1,3)$.

It seems now an appropriate time to introduce the usual notation for unit vectors in special and general relativity, and in the study of Clifford algebras. The four basis vectors, describing the coordinates of an event, or describing translations of elements of the linear space are

$$
\begin{aligned}
e_0 &= c\hat{\mathbf{t}}, && (2.49)\\
e_1 &= \hat{\mathbf{x}}, && (2.50)\\
e_2 &= \hat{\mathbf{y}}, && (2.51)\\
e_3 &= \hat{\mathbf{z}}, && (2.52)
\end{aligned}
$$

which we index by Greek letters, $0, 1, 2, 3$, and use as usual Latin letters for the spatial indices $1, 2, 3$. An event a measured in some frame S is thus

$$
\begin{aligned}
a &= a_0/c\,\hat{\mathbf{t}} + a_1\hat{\mathbf{x}} + a_2\hat{\mathbf{y}} + a_3\hat{\mathbf{z}},\\
&= a_0 e_0 + a_1 e_1 + a_2 e_2 + a_3 e_3,\\
&= a_0 e_0 + a_i e_i,\\
&= a_\mu e_\mu, && (2.53)
\end{aligned}
$$

where as usual we use non-bold Latin font for mono-vectors (vectors in spacetime). We also use the usual 'Einstein summation convention' whereby doubled indices are summed over. However because we are using a Clifford algebra, we do not need to use raised and lowered indices to take into account the metric – the metric is built into the basis vectors. The additional notation for the Clifford algebra basis elements is constructed from products of the defining elements of equations (2.49)-(2.52).

$$
\begin{aligned}
e_{\mu\nu} &\equiv e_\mu e_\nu \;= -e_\nu e_\mu \;= -e_{\nu\mu}, && (2.54)\\
e_{\mu\nu\rho} &\equiv e_\mu e_\nu e_\rho \quad \text{and cyclic permutations}, && (2.55)\\
&= -e_{\nu\mu\rho} \;\; \text{and other non-cyclic permutations}, && (2.56)\\
e &\equiv e_0 e_1 e_2 e_3 \;=\; e_{0123} \quad \text{and cyclic permutations}, && (2.57)\\
&= = -e_{1023} \quad \text{and other non-cyclic permutations}. && (2.58)
\end{aligned}
$$

We may expand out the product a^2 as

$$
\begin{aligned}
a^2 &= (a_0 e_0 + a_1 e_1 + a_2 e_2 + a_3 e_3)^2,\\
&= a_0^2 e_0^2 + (a_i e_i)^2 + a_0 a_i e_{0i} + a_i a_0 e_{i0},\\
&= a_0^2 - a_1^2 - a_2^2 - a_3^2, && (2.59)
\end{aligned}
$$

since $e_{0i} = -e_{i0}$.

Observe that the spacetime Clifford algebra $C\ell(1,3)$ contains the 16 linearly independent basis elements

$$
\begin{aligned}
\text{the scalar, } 1 &= 1, && (2.60)\\
\text{the 4 mono-vectors, } e_\mu &= e_0, e_1, e_2, e_3 \;=\; c\hat{\mathbf{t}}, \hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}, && (2.61)\\
\text{the 3 spatial bi-vectors, } e_{ij} &= e_{23}, e_{31}, e_{12} \;=\; \hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}, && (2.62)\\
\text{the 3 space-time bi-vectors, } e_{i0} &= e_{10}, e_{20}, e_{30}, && (2.63)\\
\text{the 4 tri-vectors, } e_\mu e &= e_{123}, e_{023}, e_{031}, e_{012}, && (2.64)\\
\text{the quadri-vector or pseudoscalar, } e &= e_{0123}, && (2.65)
\end{aligned}
$$

where the four elements $1, e_0, e_{10}, e_{20}, e_{30}$, and $, e_{123}$ square to $+1$ and the ten elements $e_1, e_2, e_3, e_{23}, e_{31}, e_{12}, e_{023}, e_{031}, e_{012}$ and e square to -1.

2.6 Summary

In this chapter we have motivated the use of the Clifford algebra $C\ell(1,3)$ for formulating physics in spacetime. The geometric product defined on an n-dimensional vector space gives rise to a $2n$-dimensional Clifford algebra. Starting with two dimensional geometry, it was shown using Pythagoras' theorem that in addition to a scalar and two orthonormal basis vectors, another mathematical object called a bi-vector is obtained by taking the product of the two basis vectors. This bi-vector turned out to induce rotation in the plane of the two orthonormal basis vectors.

Generalising to three spatial dimensions, rotations are no longer commutative. The eight dimensional Clifford algebra $C\ell(0,3)$ contains three bi-vectors that do not commute. These bi-vectors provide a tool for describing rotations geometrically in a plane instead of about an axis of rotation. The choice of metric (and consequently, the choice of Clifford algebra) is important. Only the anti-Euclidean metric preserves handedness.

Time can be introduced as a fourth coordinate with an associated unit vector, orthonormal to the three spatial unit vectors. The definition of speed as the distance traveled divided by the time taken, together with the constancy of the speed of light c, leads to the Lorentz metric with signature $(1,-1,-1,-1)$ and consequently the Clifford algebra $C\ell(1,3)$, which consists of 16 linearly independent basis vectors. Because the metric is built into the basis vectors, we do not need to concern ourselves with raising or lowering indices.

Chapter 3

Matrix Representation of Clifford algebras

3.1 Introduction

Matrices are a natural and very useful way to study the properties of algebras. The work presented in this chapter is an independent investigation of known results on the structure of low dimensional Clifford algebras. We find the matrix representations of various Clifford algebras, in particular the algebras $C\ell(0,3)$, $C\ell(3,0)$, $C\ell(1,3)$ and $C\ell(3,1)$ and some of their various subalgebras in two and one spatial dimensions. We consider here only the matrix representations of $C\ell(p,q)$ up to $p+q \leq 4$. For representations up to $p+q=7$ the reader is referred to Lounesto [21].

The selection of Clifford algebras we consider here reflects the algebras that were derived from the geometry of physical space in the previous chapter. In addition to these, we also consider the algebras $C\ell(1,0)$ and $C\ell(0,1)$ to highlight the isomorphism between the complex number algebra $\mathbb{C}$ and $C\ell(0,1)$. We show that although these two algebras are isomorphic, there are significant differences in the way rotations are described geometrically in these algebras. Some algebras are omitted in this chapter (for example $C\ell(2,1)$ and $C\ell(4,0)$) because they are not directly relevant to any of the material presented in this thesis.

Although matrix representations are a useful tool for studying Clifford algebras, and indeed algebras in general, it is sometimes preferable to work with the Clifford algebras themselves because the geometry is often more transparent. There are merits in both using matrices and working with the elements of the Clifford algebra.

There are two important points we must consider when looking for a matrix representation. First the dimensions of the algebra cannot be more than the number of independent components of the matrices. Second, it would be nice if the geometry found in the Clifford algebras can also be found in their matrix representations.

3.2 $C\ell(p,q)$ with $p+q=1$

Before discussing the representations of the Clifford algebras $C\ell(1,0)$ and $C\ell(0,1)$, we first consider representations of the complex number algebra. A complex number has two degrees of freedom and may be written $a+ib$ where a and b are both real. Consider a two dimensional vector space with basis

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.1}$$

The basis matrices satisfy $I_2^2 = 1, i^2 = -1$. Every complex number $a+ib$ may be written as a 2×2 matrix with real entries

$$a + ib = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \qquad a, b \in \mathbb{R}. \tag{3.2}$$

Adding and multiplying these matrices gives matrices of the same form and so we have found a matrix representation of the complex number algebra.

The Clifford algebras $C\ell(1,0)$ and $C\ell(0,1)$ each have two basis elements, 1 and e_1 satisfying

$$1^2 = 1, \quad e_1^2 = +1, \text{ if } e_1 \in C\ell(1,0), \tag{3.3}$$

$$1^2 = 1, \quad e_1^2 = -1, \text{ if } e_1 \in C\ell(0,1). \tag{3.4}$$

From the previous chapter we have that given $e_i = 1...n$, one obtains $2n$ basis multivectors which give rise to a 2^n-dimensional Clifford algebra. Thus, 1-dimensional geometry gives rise to a 2-dimensional Clifford algebra.

It is not possible to represent both these algebras by the same set of matrices used to represent the algebra of complex numbers because

$$I_2^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^2 = 1, \quad i^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = -1, \tag{3.5}$$

whereas a representation of $C\ell(1,0)$ requires a matrix representation with both basis elements squaring to unity.

To find representations of the algebras $C\ell(1,0)$ and $C\ell(0,1)$, consider the set of all 2×2 matrices. These matrices of course form a vector space and an arbitrary matrix

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ may be written as a sum of the four basis elements.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.6}$$

This basis however is not very useful for us since the basis elements do not square to plus or minus unity as do the Clifford algebra basis elements. We are better to choose the following basis

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \ell = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.7}$$

$$m = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad n = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.8}$$

as they have the property that

$$1^2 = \ell^2 = m^2 = 1, \text{ and } n^2 = -1. \tag{3.9}$$

An arbitrary 2×2 matrix can be written in this basis as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{a+d}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{a-d}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.10}$$

$$+ \frac{b+c}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{b-c}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.11}$$

There are three basis elements that square to plus unity (ℓ and m) and one basis element (n) that squares to minus unity. This provides the following matrix representations for $C\ell(1,0)$ and $C\ell(0,1)$

$$1 = I_2, \qquad e_1 = \ell, \text{ or } e_1 = m \text{ for } C\ell(1,0), \tag{3.12}$$

$$1 = I_2, \qquad e_1 = n \text{ for } C\ell(0,1). \tag{3.13}$$

The elements $w = a + be_1$ of $C\ell(1,0)$ may be represented by matrices of the form

$$a + be_1 = \begin{pmatrix} a+b & 0 \\ 0 & a-b \end{pmatrix} \text{ or } a + be_1 = \begin{pmatrix} a & b \\ b & a \end{pmatrix}. \tag{3.14}$$

We thus have two different representations in terms of 2×2 matrices with real entries.

Similarly, the elements $v = a + be_1$ of $C\ell(1,0)$ may be represented by matrices of the form

$$a + be_1 = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}. \tag{3.15}$$

This matrix representation of $C\ell(1,0)$ is also a representation of the complex number algebra we found earlier. The algebra of complex numbers $\mathbb{C}$ is isomorphic to the Clifford algebra $C\ell(0,1)$.

$$\mathbb{C} \cong C\ell(0,1). \tag{3.16}$$

Explicitly, the isomorphism is given by

$$1 = 1, \qquad i = e_1. \tag{3.17}$$

Although the two algebras are isomorphic, the description of the geometry of 1-dimensional space provided by Clifford algebra is different from the standard Argand diagram view of complex numbers where a complex number z is a point in a two dimensional plane. In the complex number algebra, z can be rotated in the complex plane. In the one dimensional geometries described by $C\ell(1,0)$ and $C\ell(0,1)$ however, there is no physical rotation operator because space is simply not big enough. Given a vector in a one dimensional space, there is no physical operation that will transform the vector into minus itself (that is, an inversion) even though mathematically such an operator could exist. More generally we say that in an n-dimensional space, an n-vector may have a mathematical inversion, but there is no geometric operation that will turn this n-vector into minus itself.

In this section we have looked at the matrix representations of the Clifford algebras $C\ell(1,0)$ and $C\ell(0,1)$ and compared it to the matrix representation of the complex number algebra. In the next sections, the matrix representations of higher dimensional Clifford algebras will be considered.

3.3 $C\ell(p,q)$ with $p+q=2$

There are three Clifford algebras $C\ell(p,q)$ that satisfy $p+q=2$, namely $C\ell(1,1)$, $C\ell(2,0)$ and $C\ell(0,2)$. Since these algebras describe geometries in two dimensional space we have to consider a set of four linearly independent matrices that satisfy the commutation relations of the basis elements $\{1, e_1, e_1, e_{12}\}$ of these algebras.

Consider again the set of all 2×2 matrices with the same basis I_2, ℓ, m and n as in the previous subsection. The matrices I_2, l, m and n satisfy the multiplication rules

$$\ell m = -m\ell = n, \quad mn = -nm = \ell, \text{ and } n\ell = -\ell n = m, \tag{3.18}$$

which are precisely the rules satisfied by the basis elements e_1, e_2 and e_{12} of $C\ell(2,0)$. Furthermore,

$$1^2 = 1 = I_2^2, \qquad e_1^2 = 1 = \ell^2, \qquad e_2^2 = 1 = m^2, \qquad e_{12}^2 = -1 = n^2, \tag{3.19}$$

and so a representation of the $C\ell(2,0)$ may be found in terms of 2×2 matrices where a multivector A in this algebra is represented by a matrix of the form

$$A = a1 + be_1 + ce_2 + de_{12} = \begin{pmatrix} a+b & c+d \\ c-d & a-b \end{pmatrix}. \tag{3.20}$$

The algebra $C\ell(1,1)$ can also be represented by the matrix algebra of 2×2 matrices $\mathrm{Mat}(2,\mathbb{R})$, because this algebra has two basis elements which square to plus unity and one basis element that squares to minus unity, like the matrices l, m, n. The basis elements of $C\ell(1,1)$ algebra satisfy

$$1^2 = 1, \quad e_1^2 = 1, \quad e_2^2 = -1, \quad e_{12}^2 = 1, \tag{3.21}$$

with

$$e_1e_2 = -e_2e_1. \tag{3.22}$$

A representation of this algebra is given by

$$1 = I_2, \quad e_1 = \ell, \quad e_2 = n, \quad e_{12} = -m. \tag{3.23}$$

A multivector A in this algebra may be written as a 2×2 matrix

$$A = a1 + be_1 + ce_2 + de_{12} = \begin{pmatrix} a+b & c-d \\ -c-d & a-b \end{pmatrix}. \tag{3.24}$$

Although both the algebras $C\ell(2,0)$ and $C\ell(1,1)$ can be represented in terms of 2×2 matrices, the geometry described by these algebras is of course different. We remind the reader of the remark made in the introduction to this chapter that geometry is very transparent in Clifford algebras. Although faithful matrix representations contain the same geometry as the Clifford algebras they represent, this geometry is often obscured and working with matrices does not easily allow one to distinguish between differences geometries.

A matrix representation of $C\ell(0,2)$ in terms of $\mathrm{Mat}(2,\mathbb{R})$ cannot be found because $\mathrm{Mat}(2,\mathbb{R})$ has only one basis element that squares to minus unity whereas in $C\ell(0,2)$ three basis elements of the algebra square to minus unity. To find a matrix representation for $C\ell(0,2)$ a set of three matrices that square to minus unity is needed. 3×3 real matrices are also too small.

We have several choices for representing $C\ell(0,2)$. We could find a representation in terms of the 2×2 matrices with *complex* entries or we could find a representation in terms of 4×4 matrices with real entries. We can also find a representation in terms of quaternions as we will see in equations (3.28)-(3.30). Mathematically $C\ell(0,2)$ is isomorphic to the quaternions $\mathbb{H}$ and may be embedded in the matrix algebras $\mathrm{Mat}(2,\mathbb{C})$ and $\mathrm{Mat}(4,\mathbb{R})$. We write

$$\mathrm{Mat}(1,\mathbb{H}) \overset{1-1}{\rightarrow} \mathrm{Mat}(2,\mathbb{C}) \overset{1-1}{\rightarrow} \mathrm{Mat}(4,\mathbb{R}). \tag{3.25}$$

A suitable set of sixteen 4×4 matrices can be constructed as tensor products of the 2×2 basis elements I_2, ℓ, m, n.

$$\begin{aligned}
A_1 &= I_2\otimes I_2 = \begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix}, \qquad A_2 = I_2\otimes \ell = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \\
A_3 &= I_2\otimes m = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \qquad A_4 = I_2\otimes n = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}, \\
A_5 &= \ell\otimes I_2 = \begin{pmatrix} \ell & 0 \\ 0 & \ell \end{pmatrix}, \qquad A_6 = \ell\otimes \ell = \begin{pmatrix} \ell & 0 \\ 0 & -\ell \end{pmatrix}, \\
A_7 &= \ell\otimes m = \begin{pmatrix} 0 & \ell \\ \ell & 0 \end{pmatrix}, \qquad A_8 = \ell\otimes n = \begin{pmatrix} 0 & \ell \\ -\ell & 0 \end{pmatrix}, \\
A_9 &= m\otimes I_2 = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}, \qquad A_{10} = m\otimes \ell = \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix}, \\
A_{11} &= m\otimes m = \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}, \qquad A_{12} = m\otimes n = \begin{pmatrix} 0 & m \\ -m & 0 \end{pmatrix}, \\
A_{13} &= n\otimes I_2 = \begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix}, \qquad A_{14} = n\otimes \ell = \begin{pmatrix} n & 0 \\ 0 & -n \end{pmatrix}, \\
A_{15} &= n\otimes m = \begin{pmatrix} 0 & n \\ n & 0 \end{pmatrix}, \qquad A_{16} = n\otimes n = \begin{pmatrix} 0 & n \\ -n & 0 \end{pmatrix}.
\end{aligned}$$

These matrices satisfy

$$\begin{aligned}
A_i^2 &= +1, \quad \text{for} \quad i = 1,2,3,5,6,7,9,10,11,16, \\
A_i^2 &= -1, \quad \text{for} \quad i = 4,8,12,13,14,15,
\end{aligned}$$

and so we have more than the required number of matrices that square to minus unity to represent $C\ell(0,2)$. One possible representation is to choose

$$e_1 = A_8 = \begin{pmatrix} 0 & \ell \\ -\ell & 0 \end{pmatrix}, \quad e_2 = A_{12} = \begin{pmatrix} 0 & m \\ -m & 0 \end{pmatrix}, \quad e_{12} = -A_{13} = -\begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix}, \tag{3.26}$$

although this choice is certainly not unique.

Finally, these matrices give a representation of the quaternion algebra $\mathbb{H}$ and so, as mentioned earlier, there exists an isomorphism between the quaternion algebra and the Clifford algebra $C\ell(0,2)$,

$$C\ell(0,2) \cong \mathbb{H}. \tag{3.27}$$

Explicitly, this 4×4 real representation of the quaternions is given by

$$i = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \tag{3.28}$$

$$j = \begin{pmatrix} 0 & n \\ n & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \tag{3.29}$$

$$k = \begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{3.30}$$

Of course we can also represent the other two algebras $C\ell(2,0)$ and $C\ell(1,1)$ using 4×4 matrices. For example, a possible representation of $C\ell(2,0)$ in $\mathrm{Mat}(4,\mathbb{R})$ is

$$e_1 = \begin{pmatrix} \ell & 0 \\ 0 & \ell \end{pmatrix}, \quad e_2 = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}, \quad e_{12} = \begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix}, \tag{3.31}$$

however this representation is just two copies of its representation in $\mathrm{Mat}(2,\mathbb{R})$.

3.4 Representations of $C\ell(0,3)$ and $C\ell(3,0)$

The algebras $C\ell(0,3)$ and $C\ell(3,0)$ describe the geometry of three space with an anti-Euclidean and Euclidean metric respectively. Both these algebras are eight dimensional containing, one scalar, three mono-vectors, three bi-vectors and one tri-vector which is the pseudoscalar. We have also seen that both algebras possess some cyclic structure. We will not concern ourselves with the matrix representation of the algebra $C\ell(1,2)$ and $C\ell(2,1)$.

Consider first the algebra $C\ell(3,0)$. The algebra has a four dimensional subalgebra consisting of the scalar and the bi-vectors $\{1, e_{23}, e_{31}, e_{12}\}$ called the even subalgebra

$C\ell^+(3,0)$ which is isomorphic to $C\ell(2,0)$ and the quaternion algebra. The isomorphism with the quaternion algebra is given by

$$1 \leftrightarrow 1, \qquad i \leftrightarrow -e_{23}, \qquad j \leftrightarrow -e_{31}, \qquad k \leftrightarrow -e_{12}, \tag{3.32}$$

but we note that this representation does not respect the cyclic structure of $e_1e_2 = e_{12}$ that we have for $C\ell(0,3)$.

$C\ell(3,0)$ has four basis elements that square to unity and four that square to minus unity. A representation of the algebra can be found in term of 4×4 real matrices,

$$1 = A_1 = \begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix}, \qquad e_1 = A_5 = \begin{pmatrix} \ell & 0 \\ 0 & \ell \end{pmatrix}, \tag{3.33}$$

$$e_2 = A_{10} = \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix}, \qquad e_3 = A_{11} = \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}. \tag{3.34}$$

From these the matrix representations of the other basis elements are readily found to be

$$e_{12} = A_{14}, \qquad e_{23} = A_4, \qquad e_{31} = -A_{15}, \qquad e_{123} = A_8. \tag{3.35}$$

It is also possible to find a representation of $C\ell(3,0)$ in terms of 2×2 complex matrices. These matrices span an eight dimensional vector space. A basis for this vector space is given by

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.36}$$

$$i = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, i\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, i\sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{3.37}$$

The matrices $\sigma_1, \sigma_2, \sigma_3$ are of course the Pauli matrices which give a representation of the Pauli algebra

$$[\sigma_a, \sigma_b] = 2i\epsilon_{abc}\sigma_c. \tag{3.38}$$

We find a representation of $C\ell(3,0)$ by choosing

$$e_1 = \sigma_1, \qquad e_2 = \sigma_2, \qquad e_3 = \sigma_3. \tag{3.39}$$

From these it is then easy to show that

$$e_{12} = \sigma_1\sigma_2 = i\sigma_3, \tag{3.40}$$

$$e_{23} = \sigma_2\sigma_3 = i\sigma_1, \tag{3.41}$$

$$e_{31} = \sigma_3\sigma_1 = i\sigma_2, \tag{3.42}$$

$$e_{123} = \sigma_1\sigma_2\sigma_3 = i. \tag{3.43}$$

Turning our attention now to the algebra $C\ell(0,3)$, this algebra has six of its eight basis multivectors squaring to minus unity and so a matrix representation in terms of 4×4 real matrices (or 2×2 complex matrices) is not possible. To find a real matrix representation we need to go to 8×8 matrices.

A suitable representation can be found in terms of the 2×2 matrix algebra over the quaternions. Let

$$1 = I_2, \qquad e_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix}, \qquad e_3 = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}. \tag{3.44}$$

From this it is easy to show that

$$e_{23} = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, e_{31} = \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix}, e_{12} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}, e_{123} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \tag{3.45}$$

Using equations (3.28)-(3.30), these matrices can be rewritten as 8×8 real matrices since i, j, k can be given as 4×4 real matrices.

3.5 Representations of $C\ell(1,3)$ and $C\ell(3,1)$

In this section we discuss the matrix representation of the algebras $C\ell(1,3)$ and $C\ell(3,1)$. Both are sixteen dimensional algebras that describe 4-dimensional geometry. The matrix representations of these two algebras are quite distinct and they have different numbers of roots of $+1$ and -1.

$$C\ell(1,3): \quad 10 \quad \text{roots of } +1, \tag{3.46}$$

$$6 \quad \text{roots of } -1, \tag{3.47}$$

$$C\ell(3,1): \quad 6 \quad \text{roots of } +1, \tag{3.48}$$

$$10 \quad \text{roots of } -1. \tag{3.49}$$

Because $C\ell(3,1)$ has only six roots of -1 we can find a representation in terms of 4×4 real matrices, since earlier we found a basis for 4×4 real matrices which has exactly six basis elements which square to minus unity and ten which square to plus unity. A suitable representation is [1]:

$$1 = \begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix}, \quad e_0 = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}, \tag{3.50}$$

[1]The author wishes to acknowledge Martin van der Mark and John Williamson for providing this specific representation in their notes

$$e_1 = \begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix}, \quad e_2 = \begin{pmatrix} \ell & 0 \\ 0 & -\ell \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & \ell \\ \ell & 0 \end{pmatrix}, \tag{3.51}$$

$$e_{10} = \begin{pmatrix} \ell & 0 \\ 0 & \ell \end{pmatrix}, \quad e_{20} = \begin{pmatrix} -n & 0 \\ 0 & n \end{pmatrix}, \quad e_{30} = \begin{pmatrix} 0 & -n \\ -n & 0 \end{pmatrix}, \tag{3.52}$$

$$e_{23} = \begin{pmatrix} 0 & \ell \\ -\ell & 0 \end{pmatrix}, \quad e_{31} = \begin{pmatrix} 0 & -m \\ -m & 0 \end{pmatrix}, \quad e_{12} = \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix}, \tag{3.53}$$

$$e_{023} = \begin{pmatrix} 0 & m \\ -m & 0 \end{pmatrix}, \quad e_{031} = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \quad e_{012} = \begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix}, \tag{3.54}$$

$$e_{123} = \begin{pmatrix} 0 & n \\ -n & 0 \end{pmatrix}, \quad e_{0123} = e = \begin{pmatrix} 0 & -\ell \\ \ell & 0 \end{pmatrix}. \tag{3.55}$$

Because $C\ell(1,3)$ has ten roots of minus unity, a representation of this algebra in terms of 4×4 matrices with real entries cannot be found. Complexifying this algebra to the 4×4 matrices with complex entries increases the number of dimensions in the parameter space to 32. $C\ell(1,3)$ is isomorphic to one of the subalgebras of $\mathrm{Mat}(4,\mathbb{C})$. The set of 4×4 matrices with complex entries is therefore large enough to accommodate $C\ell(1,3)$. We however prefer to find a real matrix representation.

The sixteen dimension algebra $C\ell(1,3)$ may be represented by 2×2 matrices with quaternion entries, that is $\mathrm{Mat}(2,\mathbb{H})$. A well known specific representation of these matrices is given by Leary [22] as

$$1 = \begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix}, \quad e_0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \tag{3.56}$$

$$e_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}, \tag{3.57}$$

$$e_{10} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad e_{20} = \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}, \quad e_{30} = \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}, \tag{3.58}$$

$$e_{23} = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \quad e_{31} = \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix}, \quad e_{12} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}, \tag{3.59}$$

$$e_{023} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad e_{031} = \begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix}, \quad e_{012} = \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix}, \tag{3.60}$$

$$e_{123} = \begin{pmatrix} 0 & -I_2 \\ -I_2 & 0 \end{pmatrix}, \quad e_{0123} = e = \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix}. \tag{3.61}$$

Again, using equations (3.28)-(3.30), these can be rewritten as 4×4 complex matrices or 8×8 real matrices.

$$\mathrm{Mat}(2, \mathbb{H}) \overset{1-1}{\to} \mathrm{Mat}(4, \mathbb{C}) \overset{1-1}{\to} \mathrm{Mat}(8, \mathbb{R}), \tag{3.62}$$

or more generally

$$\mathrm{Mat}(n, \mathbb{H}) \overset{1-1}{\to} \mathrm{Mat}(2n, \mathbb{C}) \overset{1-1}{\to} \mathrm{Mat}(4n, \mathbb{R}), \tag{3.63}$$

for some positive integer n.

Note that the above representation explicitly highlights the link of the quaternions with the bi-vectors e_{ij} and the Pauli spin matrices with the bi-vectors e_{0i}. Rotations can therefore be given an acceptable treatment in any of these matrix algebras.

We conclude this section with a table summarising the matrix representation of the Clifford algebras considered in this chapter. We remind the reader that the table does not list all matrix representations. Other representations may be found, for example via equation (3.63).

Clifford algebra	Matrix representation	Clifford algebra	Matrix representation
$C\ell(0,1)$	$\mathrm{Mat}(1,\mathbb{C})$	$C\ell(0,3)$	$\mathrm{Mat}(2,\mathbb{C})$
$C\ell(1,0)$	$\mathrm{Mat}(2,\mathbb{R})$	$C\ell(3,0)$	$\mathrm{Mat}(2,\mathbb{H})$
$C\ell(0,2)$	$\mathrm{Mat}(1,\mathbb{H})$	$C\ell(1,3)$	$\mathrm{Mat}(2,\mathbb{H})$
$C\ell(2,0)$	$\mathrm{Mat}(2,\mathbb{R})$	$C\ell(3,1)$	$\mathrm{Mat}(2,\mathbb{C})$
$C\ell(1,1)$	$\mathrm{Mat}(2,\mathbb{R})$		

Table 3.1: Possible matrix representations of some Clifford algebras. Other matrix representations are possible. The representations listed here are those given throughout the chapter.

3.6 Summary

In this independent investigation on the structure of Clifford algebras we have found matrix representations for several important algebras. Some algebras ($C\ell(1,2)$, $C\ell(2,1)$ and $C\ell(2,2)$) were not covered here for the reason that they are not relevant to this thesis.

Matrices are a natural and useful way of studying various properties of algebras. Although working with matrices has many advantages, one disadvantage is that the geometry is often obscured. Working with matrix algebras does not always let us distinguish between different geometries easily. This is not a problem encountered when using Clifford algebras because the geometry is very transparent.

The representation of $C\ell(1,3)$ in terms of $\mathrm{Mat}(2,\mathbb{H})$ highlights the link between the quaternions and the bi-vectors e_{ij} and the Pauli spin matrices and the bi-vectors e_{0i}. Rotations can be given an acceptable treatment in any of these algebras.

Chapter 4

Division in the Clifford algebra of spacetime

4.1 Introduction

One claimed weakness [1] of the Clifford algebra $C\ell(1,3)$ in being able to mathematically describe reality is that the algebra is not a division algebra. Many algebras including the algebra of the reals, the complex numbers and the quaternion algebra are division algebras, meaning that the only element of the algebra for which no inverse can be found is the element 0. For the Clifford algebra $C\ell(1,3)$ this is not the case and there exist many elements A for which no inverse A^{-1} can be defined in such a way that $AA^{-1} = 1$ can be found. It is not hard to imagine that this could potentially cause serious problems for the algebra in its success to describe spacetime physics.

In this chapter we show that the lack of division in the algebra is not an actual weakness of the spacetime algebra but that it can in fact be argued that it is necessary. We outline two reasons why it is important to find when a multivector is invertible and when it is not.

In section 4.3 we define the Clifford group $\Gamma^{1,3}$ associated with the spacetime algebra $C\ell(1,3)$. This group is grade preserving. That is: a single grade multivector is mapped to a multivector of the same grade under the action of the group, $\Gamma^{1,3} : V_k \to V_k$, where V_k is the k-grade subspace of the Clifford algebra[1]. Maxwell's equations can be written in $C\ell(1,3)$ in terms of a mono-vector potential α. In chapter 6 it will be shown that an extended set of equations called the generalised Maxwell equations are written not in terms of a mono-vector potential but rather in terms of an odd vector (that is, mono-vector plus tri-vector) potential $\alpha + \beta e$, where e is the pseudoscalar. Furthermore, in the

[1]more on this in section 4.3

usual Clifford algebra formulations of the Heisenberg and Poincaré algebras the position and momentum operators can be written as mono-vectors and tri-vectors respectively[2]. As we shall see in chapter 8, a Clifford algebraic representation of the stabilised Poincaré-Heisenberg algebra requires that both the position and momentum operators be written as general odd vectors in $C\ell(1,3)$ (that is, mono-vector plus tri-vector) instead of mono-vectors and tri-vectors respectively.

For such cases, the Clifford group is too restrictive and of limited use. Instead of a grade preserving group, what we want is a group that preserves only the even or oddness of a multivector under the group action, namely preserves parity. We give a definition of parity in section 4.3. Under this new group a mono-vector can be mapped to a mono-vector plus tri-vector for example. It will be shown in section 4.3 that this group consists of all invertible elements $g \in C\ell(1,3)$ where g is either even or odd. It is therefore very important to know which even or odd multivectors in $C\ell(1,3)$ are invertible and which are not.

It has been shown by van der Mark and Williamson [10] that the areas of the algebra where division cannot be defined correspond to the areas where certain invariant quantities become zero, for example on the light cone. The areas where division is undefined are referred to as null-hyperplanes because they correspond to null multivectors. These null-hyperplanes correspond exactly to the limiting cases of physical interest. The fact that there does not exist an inverse for every element is therefore not a weakness but necessary because the breakdown of division in these areas matches the behavior of nature.

In section 4.4, we confirm some of the results found in [10], but not by means of defining a new conjugate, but by using the matrix representations of the spacetime Clifford algebra $C\ell(1,3)$. The use of matrix representations make it a straightforward task to determine which elements of the algebra are and are not invertible. Given that an element is invertible, it is straightforward to calculate its inverse.

The purpose of finding when a multivector can be inverted is thus twofold. First, it is important to know what even and odd multivectors are invertible to determine what elements belong to the extended Clifford group. This group could play an important role in the generalised Maxwell equations and stabilised Poincaré-Heisenberg algebra. Second, the invertibility or non-invertibility of multivectors gives physical insight into conserved quantities and limitations of physical systems.

[2]It should be noted that there is some freedom in how to represent the momentum operators in the Clifford algebra. It makes sense to start with writing the position operator as a mono-vector. This then allows at least two possibilities for representing the momentum vector, one as a tri-vector and one as a scalar plus bi-vector. The choice here reflects the author's preference.

4.2 Finding inverses in the spacetime algebra $C\ell(1,3)$

It may be recalled from chapter 3 that the there exists a representation of the spacetime algebra $C\ell(1,3)$ in terms of 2×2 matrices with quaternion entries. In this section we suggest that the easiest method to find inverses of multivectors in the algebra is by making use of this matrix representation. Such an approach avoids the introduction of a new conjugate operator as in [10]. Of course, a matrix fails to be invertible when the determinant is equal to zero. We can apply this condition to the matrix representation of the algebra $C\ell(1,3)$ and in this way find what multivectors are invertible and which are not. Care must be taken however since the quaternions themselves are represented by 4×4 real matrices or 2×2 complex matrices and so $\mathrm{Mat}(2,\mathbb{H})$ matrices are really 8×8 matrices but written in block form. It is however easy to obtain an expression for the determinant of a matrix written in block form.

Consider a matrix A written in block form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \qquad \text{where } A_{ij} \text{ are square matrices.} \tag{4.1}$$

The determinant of such a matrix may be expressed as[3]

$$\begin{aligned} \det(A) &= \det(A_{11})\det(A_{22} - A_{21}A_{11}^{-1}A_{12}), & (4.2)\\ &= \det(A_{22})\det(A_{11} - A_{12}A_{22}^{-1}A_{21}). & (4.3) \end{aligned}$$

An arbitrary multivector $\Omega \in C\ell(1,3)$ can be written in terms of the $\mathrm{Mat}(2,\mathbb{H})$ representation as

$$\Omega = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}, \qquad \text{where } q_{ij} \text{ are quaternions.} \tag{4.4}$$

The determinant of a quaternion q is given by $\det(q) = |q|^2$ and so the determinant of the multivector $\Omega \in C\ell(1,3)$ is

$$\begin{aligned} \det(\Omega) &= |q_{11}|^2|q_{22} - q_{21}q_{11}^{-1}q_{12}|^2, \qquad (q_{11} \neq 0), & (4.5)\\ &= |q_{22}|^2|q_{11} - q_{12}q_{22}^{-1}q_{21}|^2, \qquad (q_{22} \neq 0). & (4.6) \end{aligned}$$

Ω is singular if and only if $\det(\Omega) = 0$, which is equivalent to saying that either $q_{11} = 0$ or

$$q_{22} = q_{21}q_{11}^{-1}q_{12} \;\text{ or }\; q_{22}q_{12}^{-1} = q_{21}q_{11}^{-1}. \tag{4.7}$$

[3] see for example the website http://en.wikipedia.org/wiki/Determinant

Equation (4.7) is the condition that determines if a given multivector Ω has an inverse or not. Provided an inverse does exist, it is straightforward to write down a formula for the inverse Ω^{-1} in terms of the 2×2 matrix representation with quaternion entries.

$$\Omega^{-1} = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}^{-1} = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}, \tag{4.8}$$

where

$$\begin{aligned} w_{11} &= (q_{11} - q_{12} q_{22}^{-1} q_{21})^{-1}, & (4.9) \\ w_{12} &= q_{11}^{-1} q_{12} (q_{21} q_{11}^{-1} q_{12} - q_{22})^{-1}, & (4.10) \\ w_{21} &= (q_{21} q_{11}^{-1} q_{12} - q_{22})^{-1} q_{21} q_{11}^{-1}, & (4.11) \\ w_{22} &= (q_{22} - q_{21} q_{11}^{-1} q_{12})^{-1}. & (4.12) \end{aligned}$$

It is thus straightforward (although perhaps tedious) to find the determinants and when possible the inverses of multivectors in the algebra $C\ell(1,3)$. In section 4.4 we use the above approach to find when some multivectors fail to have an inverse.

4.3 The extended Clifford group

The algebra $C\ell(1,3)$ can be decomposed as a direct sum of subspaces

$$C\ell(1,3) = \bigoplus_{k=0}^{3} V_k = V_0 \bigoplus V_1 \bigoplus V_2 \bigoplus V_3 \bigoplus V_4. \tag{4.13}$$

Furthermore, $C\ell(1,3)$ may also be written as the direct sum of even and odd *parity* terms

$$C\ell(1,3) = C\ell^+(1,3) \bigoplus C\ell^-(1,3), \tag{4.14}$$

where

$$C\ell^+(1,3) = \bigoplus_{k \text{ even}} V_k, \quad \text{and} \quad C\ell^-(1,3) = \bigoplus_{k \text{ odd}} V_k. \tag{4.15}$$

The even parity terms of a multivector are those terms belonging to $C\ell^+(1,3)$. Similarly, the odd parity terms are those belonging to $C\ell^-(1,3)$. It should be noted that $C\ell^+(1,3)$ is a subalgebra of $C\ell(1,3)$ but $C\ell^-(1,3)$ is not. This is because the product of two odd vectors yield an even vector under the geometric product.

An element of $C\ell(1,3)$ is said to be *homogeneous* if it has either even or odd parity. So

$$u \text{ homogeneous } \Leftrightarrow u \in C\ell^+(1,3) \quad \text{or} \quad C\ell^-(1,3). \tag{4.16}$$

With these preliminary definitions, we now define the Clifford group.

We define the Clifford group $\Gamma^{1,3}$ of the spacetime algebra $C\ell(1,3)$ to be the group of homogeneous invertible elements g such that the map

$$\pi_g : x \to gxg^{-1} \qquad x \in \mathbb{R}^{1,3}, \tag{4.17}$$

is a map from the vector space $\mathbb{R}^{1,3}$ to itself. That is, $\pi_g : V_1 \to V_1$ preserves 1-grade.

It can be shown by induction that the map π_g preserves grade in general[4]. That is, if $g \in \Gamma^{1,3}$, then $\pi_g : V_k \to V_k$ is a bijection for all $k = 0..4$. The result is true by definition for $k = 1$ and trivially too for $k = 0$. Suppose the result is true up to some integer $k \geq 1$.

Let $g \in \Gamma^{p,q}$ and $u = e_{\mu_1\mu_2\ldots\mu_{k+1}}$. It suffices to show that $\pi_g u \in V_{k+1}$. We have

$$\pi_g(u) = \pi_g(e_{\mu_1\mu_2\ldots\mu_k})\pi_g(e_{\mu_{k+1}}) \in V_{k+1} + V_{k-1}. \tag{4.18}$$

Let $v \in V_{k-1}$. It suffices to show that

$$\langle \pi_g(u), v \rangle = 0, \tag{4.19}$$

where $\langle a, b \rangle = \langle ab \rangle_0\,,\ a, b \in C\ell(1,3)$ is an inner product. Since π_g is an onto map (by assumption) from $V_{k-1} \to V_{k-1}$, $v = \pi_g(x)$ for some $x \in V_{k-1}$. But then

$$\langle \pi_g(u), v \rangle = \langle \pi_g(u), \pi_g(x) \rangle = \langle u, x \rangle = 0, \tag{4.20}$$

and the result follows. Although in this chapter we are only considering $\Gamma^{1,3}$, the same proof holds for any $\Gamma^{p,q}$.

There is an important anti-automorphism defined on $C\ell(1,3)$[5] called *reversion* which satisfies

$$\tilde{a} = a \qquad a \in \mathbb{R}^{1,3}, \tag{4.21}$$

$$\widetilde{ab} = \tilde{b}\tilde{a} \qquad a, b \in C\ell(1,3). \tag{4.22}$$

A reversion reverses the order of all the indices of the basis elements of a Clifford algebra. For a general element $a \in C\ell(1,3)$ we have

$$\tilde{a} = \langle a \rangle_0 + \langle a \rangle_1 - \langle a \rangle_2 - \langle a \rangle_3 + \langle a \rangle_4\,, \tag{4.23}$$

where $\langle a \rangle_k$ refers to the k-grade component of a.

[4]the author acknowledges Peter Renaud for providing the following proof.

[5]The definition is easily generalised to $C\ell(p,q)$

The relationship between the Clifford group $\Gamma^{1,3}$ and various (s)pin groups is given by Lounesto [21] as follows

$$\begin{aligned} \mathrm{Pin}(1,3) &= \left\{ g \in \Gamma^{1,3} : g^{-1} = \pm\tilde{g} \right\}, & (4.24) \\ \mathrm{Spin}(1,3) &= \mathrm{Pin}^{+}(1,3), & (4.25) \\ \mathrm{Spin}^{\uparrow}(1,3) &= \left\{ g \in \mathrm{Pin}^{+}(1,3) : g^{-1} = +\tilde{g} \right\}. & (4.26) \end{aligned}$$

The spin group is a $2:1$ covering group of the group $SO(1,3)$. Also, $\mathrm{Spin}^{\uparrow}(1,3)$ is a covering group of the proper Lorentz group $\mathrm{SL}(2,\mathbb{C})$.

There are situations where the conditions placed on g by the Clifford group are too restrictive. For example, Maxwell's equations can be written in terms of a mono-vector potential α. In chapter 6 it will be shown that an extended set of equations called the generalised Maxwell equations are written not in terms of a mono-vector potential but rather in terms of a mono-vector plus tri-vector potential $\alpha + \beta e$. Furthermore, in the usual Clifford algebra formulation of the Heisenberg and Poincaré algebras the position and momentum operators are written as mono-vectors and tri-vectors respectively. As we shall see in chapter 8, a Clifford algebraic representation of the stabilised Poincaré-Heisenberg algebra requires that both the position and momentum operators be written as general odd vectors in $C\ell(1,3)$ (that is, mono-vector plus tri-vector). For such cases, the Clifford group is too restrictive since it does not mix grades.

What we would like is a group of invertible elements which do not preserve the grade but rather preserve the parity of multivectors. Proposition 4.1 of [28] states that if $n = p + q$ is even, as for $C\ell(1,3)$, and if the inner automorphism

$$\pi_g : x \to gxg^{-1}, \tag{4.27}$$

preserves parity, then g must be homogeneous. It is readily shown that the converse of this statements also holds. That is; if g is homogeneous, then the inner automorphism π_g preserves parity. This means that the group we seek is the group of all homogeneous elements of $C\ell(1,3)$ for which an inverse exists. We call this group the *extended* Clifford group and write $\Gamma^{1,3}_{\mathrm{ext}}$.

The Clifford group $\Gamma^{1,3}$ is a subgroup of this extended Clifford group $\Gamma^{1,3}_{\mathrm{ext}}$

$$\Gamma^{1,3} \subset \Gamma^{1,3}_{\mathrm{ext}}. \tag{4.28}$$

To determine what homogeneous elements of $C\ell(1,3)$ are invertible, we use the matrix representation of section 3.5 to write the homogeneous elements as 2×2 matrices and use equations (4.5) and (4.6) to calculate their determinants. For any homogeneous element g in $C\ell(1,3)$, either $g \in C\ell^{+}(1,3)$ in which case g can be written $g = q_1 + q_2 e$ where $q_1, q_2 \in \mathbb{H}$

or else $g \in C\ell^-(1,3)$ in which case g can be written $g = e_0 u$ where $u \in C\ell^+(1,3)$. Because $\det(e_0 u) = \det(e_0)\det(u)$ and $\det(e_0) = 1$[6], regardless of whether g is even or odd we have to find when

$$q_1 + q_2 e = q_1 \begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix} + q_2 \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix} \tag{4.29}$$

$$= \begin{pmatrix} q_1 & -q_2 \\ q_2 & q_1 \end{pmatrix}, \tag{4.30}$$

has an inverse to determine which homogeneous elements g belong to the extended Clifford group.

We now want to find the covering groups of the extended Clifford group $\Gamma^{1,3}_{\text{ext}}$. The reader is reminded that the extended Clifford group consists of all invertible homogeneous elements of $C\ell(1,3)$.

Suppose $\alpha \in C\ell(1,3)$ is both homogeneous and even. We can write

$$\alpha = q_1 + q_2 e. \tag{4.31}$$

Taking the reversion of α, we get

$$\tilde{\alpha} = \tilde{q}_1 + \tilde{e}\tilde{q}_2, \tag{4.32}$$

$$= \overline{q}_1 + \overline{q}_2 e, \tag{4.33}$$

where the conjugate $\overline{q}$ of q is defined as changing the sign of the non-scalar terms of q while leaving the scalar term unchanged.

Now

$$\alpha\tilde{\alpha} = (q_1 + q_2 e)(\overline{q}_1 + \overline{q}_2 e), \tag{4.34}$$

$$= (|q_1|^2 - |q_2|^2) + (q_1\overline{q}_2 + q_2\overline{q}_1)e, \tag{4.35}$$

$$= (|q_1|^2 - |q_2|^2) + 2\langle q_1\overline{q}_2\rangle_0 e. \tag{4.36}$$

Therefore, $\alpha\tilde{\alpha}$ is of the form $x + ye$ where $x, y \in \mathbb{R}$.

Choose $h = a + be$ a,b real such that

$$h^2 = \alpha\tilde{\alpha}, \tag{4.37}$$

and let

$$u = \alpha h^{-1}. \tag{4.38}$$

[6] This is easily checked using the matrix representation of section 3.5.

We now have

$$\begin{aligned} u\tilde{u} &= (\alpha h^{-1})(h^{-1}\tilde{\alpha}), && (4.39)\\ &= h^{-2}\alpha\tilde{\alpha} = 1. && (4.40) \end{aligned}$$

This means that $u \in \mathrm{Spin}^{\uparrow}(1,3)$. We can now decompose α as follows

$$\alpha = hu = (a + be)u. \qquad (4.41)$$

Any even element in $C\ell(1,3)$ can be written as the product of a bi-vector in $\mathrm{Spin}^{\uparrow}(1,3)$ and an element $(a + be)$

$$\alpha = (a + be)u,\ u \in \mathrm{Spin}^{\uparrow}(1,3) \qquad \forall\alpha \in \Gamma^{1,3}_{\text{ext}}. \qquad (4.42)$$

The above calculations may be repeated for an odd parity element $\beta \in C\ell(1,3)$ to obtain a similar result[7]. Any element of the extended Clifford group can be written as the product of an element of the $\mathrm{Spin}^{\uparrow}(1,3)$ group and an element $(a + be)$. Because multiplying h (and consequently α) by a constant does not change the inner automorphism (4.27), we can normalise h such that $a^2 + b^2 = 1$. The structure of the extended Clifford group is then $\mathrm{Spin}^{\uparrow}(1,3) \times \mathrm{U}(1)$, or $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{U}(1)$ in the usual physicists notation.

4.4 The non-invertible elements of $C\ell(1,3)$

The purpose of this section is to look at some specific multivectors and find when no inverse exists for these multivectors. Only homogeneous multivectors are considered here because the extended Clifford group contains all such multivectors that are invertible. In particular, we consider some mono-vectors, a bi-vector and finally a general even parity multivector. A more complete treatment of what follows can be found in [10]. There, multivectors that are not homogeneous are also considered.

4.4.1 Mono-vectors

As a first example consider a general mono-vector in $C\ell(1,3)$. We write $x = (x_0, \mathbf{x}) = x_0e_0 + x_1e_1 + x_2e_2 + x_3e_3$. In terms of the $\mathrm{Mat}(2,\mathbb{H})$ representation of section 3.5 this vector is written as

$$\begin{aligned} x &= x_0\begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} + x_1\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + x_2\begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix} + x_3\begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}, \\ &= \begin{pmatrix} x_0 & P \\ P & -x_0 \end{pmatrix}, && (4.43) \end{aligned}$$

[7] The calculations are easily repeated by writing $\beta = e_0\alpha$.

where $P = P_1 i + P_2 j + P_3 k$ is a pure quaternion. The determinant of this vector is given by

$$\det(x) = |x_0|^2| - x_0 - P x_0^{-1} P|^2, \tag{4.44}$$

and so x fails to have an inverse if and only if

$$x_0^2 - x_1^2 - x_2^2 - x_3^3 = 0. \tag{4.45}$$

For the case where x is a position vector in spacetime, equation (4.45) is of course equal to the invariant interval x^2. From relativity we know that this interval (4.45) being zero, corresponds to being on the lightcone. The plane where division is not defined for mono-vectors is precisely in agreement with physical limitations set in place by the speed of light.

As another example of a mono-vector, let us consider the differential operator d,

$$d = \frac{\partial}{\partial x_\mu e_\mu} = \partial_0 e_0 - \partial_1 e_1 - \partial_2 e_2 - \partial_3 e_3. \tag{4.46}$$

Therefore, given some function f, df does not have an inverse when

$$\Box f = (\partial_0^2 - \nabla^2) f = 0. \tag{4.47}$$

Similarly, consider the vector potential $A = (\phi, \mathbf{A}) = \phi e_0 + A_1 e_1 + A_2 e_2 + A_3 e_3$. This potential does not have an inverse when

$$\phi^2 = |\mathbf{A}|^2. \tag{4.48}$$

Via Lorentz transformation it is always possible to find a frame where $A_2 = A_3 = 0$ such that $|\mathbf{A}|^2 = |A_1|^2$. In this frame, the potential A does not have an inverse if $\phi = \pm|A_1|$.

4.4.2 Bi-vector

Next, consider any bi-vector $F \in C\ell(1,3)$. In terms of the 2×2 matrix representation of section 3.5, F is written as

$$F = \begin{pmatrix} P_1 & -P_2 \\ P_2 & P_1 \end{pmatrix}, \tag{4.49}$$

where P_1 and P_2 are both pure quaternions. For an arbitrary pure quaternion P, $P^2 = -|P|^2$ and therefore the inverse of P is given by

$$P^{-1} = -\frac{P}{|P|^2}. \tag{4.50}$$

Using equations (4.5) and (4.6), the bi-vector F does not have an inverse if

$$\begin{aligned} P_1 &= P_2 P_1^{-1} P_2, &(4.51)\\ \Rightarrow P_1 &= P_2 \frac{P_1}{|P_1|^2} P_2. &(4.52) \end{aligned}$$

This condition implies that $|P_1| = |P_2|$,

$$\begin{aligned} P_2 P_1 &= P_2^2 \frac{P_1}{|P_1|^2} P_2, &(4.53)\\ &= -\frac{|P_2|^2}{|P_1|^2} P_1 P_2, &(4.54)\\ &= \frac{|P_2|^2}{|P_1|^2} P_2 P_1, &(4.55)\\ &\Rightarrow |P_1| = |P_2|. &(4.56) \end{aligned}$$

The two pure quaternions can therefore be written as

$$\begin{aligned} P_1 &= |P_1|\hat{P}_1, \qquad \hat{P}_1^2 = 1,\\ P_2 &= |P_2|\hat{P}_2, \qquad \hat{P}_2^2 = 1. \end{aligned}$$

Because $|P_1| = |P_2|$, F does not have an inverse if

$$\begin{aligned} \hat{P}_1 &= \hat{P}_2 \hat{P}_1 \hat{P}_2, &(4.57)\\ \Rightarrow \hat{P}_1 \hat{P}_2 &= -\hat{P}_2 \hat{P}_1, &(4.58)\\ \Rightarrow \hat{P}_1 \cdot \hat{P}_2 &= 0. &(4.59) \end{aligned}$$

Thus, F singular implies that

$$|P_1| = |P_2|, \text{ and } P_1 \perp P_2. \tag{4.60}$$

It will be shown in chapter 6 that the electromagnetic field is written as a bi-vector in $C\ell(1,3)$. Explicitly,

$$F = E_1 e_{01} + E_2 e_{02} + E_3 e_{03} + B_1 e_{23} + B_2 e_{31} + B_3 e_{12}, \tag{4.61}$$

where E_i and B_i are the electric and magnetic field components respectively.

The reader is reminded that the spatial bi-vectors e_{ij} are isomorphic to the pure quaternions. We substitute

$$P_1 = \mathbf{B}, \qquad P_2 = -e\mathbf{E}, \tag{4.62}$$

where $\mathbf{E} = (E_1, E_2, E_3)$ and $\mathbf{B} = (B_1, B_2, B_3)$ are the electric and magnetic Heaviside-Gibbs field vectors and $e = e_{0123}$ is the pseudoscalar. The lack of an inverse then implies that

$$\mathbf{E}^2 = \mathbf{B}^2, \quad \text{and} \quad \mathbf{E} \perp \mathbf{B}, \tag{4.63}$$

that is, free electromagnetic waves.

4.4.3 Even vector

More general than the bi-vector F, is an even vector $G \in C\ell^+(1,3)$. G can be written as

$$G = q_1 + q_2 e = \begin{pmatrix} q_1 & -q_2 \\ q_2 & q_1 \end{pmatrix}, \tag{4.64}$$

where q_1,q_2 are quaternions.

Again using equations (4.5) and (4.6), G does not have an inverse when

$$\begin{aligned} q_1 &= -q_2 q_1^{-1} q_2, & (4.65) \\ &= -q_2 \frac{q_1}{|q_1|^2} q_2. & (4.66) \end{aligned}$$

Again, by similar reasoning as for the bi-vector case, the above condition implies that

$$|q_1| = |q_2|. \tag{4.67}$$

Writing $q_1 = |q_1|u_1$ and $q_2 = |q_2|u_2$ where u_1, u_2 are unit quaternions, the condition (4.65) may now be written in terms of the conjugate $\overline{u}$ of u

$$\begin{aligned} u_1 &= -u_2 \overline{u_1} u_2, & (4.68) \\ \Rightarrow u_1 \overline{u_2} &= -u_2 \overline{u_1}, & (4.69) \\ &= -\overline{(u_1 \overline{u_2})}. & (4.70) \end{aligned}$$

Let $u_1 \overline{u_2} = l$. Then $l = -\bar{l}$ and consequently, l must be a pure quaternion. u_1 can now be written in terms of u_1 and l

$$u_1 = l u_2. \tag{4.71}$$

Because u_1 and u_2 are both unit, it follows that l is unit also.

At this stage it is not clear to the author what this condition means physically and geometrically and we leave it as an open problem. In light of chapter 6, an even vector gives rise to the fields of the generalised Maxwell equations. Perhaps then, the above condition gives us a generalisation of the conditions we found for the bi-vector case (electric and magnetic fields) in the previous subsection. If so then it would suggest that for the generalised Maxwell equations, the electric and magnetic fields are no longer necessarily normal to one another and equal in magnitude. Also, the eight dimensions of an even vector are enough to describe spin one half particles. Plane wave solutions to the Dirac equation should therefore satisfy this condition also. More work is required here.

4.5 Summary

The spacetime Clifford algebra $C\ell(1,3)$ is not a division algebra. This has led the author of reference [1], and probably others also, to reason that the algebra is therefore not a suitable mathematical structure to model physical reality. The existence of inverses is indeed very important.

In this chapter, we have highlighted two reasons why knowing when elements of the spacetime Clifford algebra are and are not invertible is important.

First, we defined a new group called the extended Clifford group which does not preserve grade but only parity. The Clifford group is a subgroup of the extended subgroup. It was found that all the elements of this group are precisely those elements of the spacetime algebra that are invertible and homogeneous.

Second, we have confirmed the observations made in [10] that the areas of the algebra where division is not defined correspond exactly to the limiting cases of physical interest, such as on the lightcone. Therefore, the behaviour of the algebra is in harmony with the behaviour of our physical universe.

We conclude therefore that the lack of division throughout the entire algebra is not to be regarded as a weakness of the algebra but necessary, since it matches the behaviour of our physical universe.

The extended Clifford group of spacetime is less restrictive than the Clifford algebra and may be more appropriate to describe the symmetries of the generalised Maxwell equations and the stabilised Poincaré-Heisenberg algebra to be discussed in chapters 6 and 8 of this thesis. It was shown that any element of the extended Clifford group of spacetime can be written as the product of an element of the Lorentz group $\mathrm{SL}(2,\mathbb{C})$ and the unitary group $\mathrm{U}(1)$.

Chapter 5

The Lorentz force from energy considerations

5.1 Introduction

We show that the Lorentz force law can be derived from the energy density of the electric and magnetic fields, together with considerations of the conservation of energy. It is often stated in the literature that the Lorentz force law is a separate yet essential supplement to Maxwell's equations (see for example page 3 of reference [29] or page 782 of [30]). It is shown that supplementing the usual expression for the energy density of the electromagnetic field with Hamilton's principle is sufficient to derive an expression for electromagnetic force. That is, the Lorentz force law.

To the best of our knowledge our derivation is novel and the result implies a significant change to the interpretation of electric and magnetic fields. However this chapter is restricted to the simple derivation of the result, not its consequences.

We first consider the static situation of the fields due to two small charged spheres, q_1 at $\mathbf{r}_1$, and q_2 at $\mathbf{r}_2$. The energy density $u(\mathbf{r})$ of the combined field at a point $\mathbf{r}$, $\mathbf{E}_1(\mathbf{r}) + \mathbf{E}_2(\mathbf{r})$, is

$$u(\mathbf{r}) = \frac{1}{2}\epsilon_0(\mathbf{E}_1(\mathbf{r}) + \mathbf{E}_2(\mathbf{r}))^2. \tag{5.1}$$

Observe that the value of $u(\mathbf{r})$ changes whenever the location of either of the two charges changes.

The total energy of the system U_{12}, being the integral over all space of $u(\mathbf{r})$, may be calculated as

$$U_{12} = \int_{\text{all space}} u(\mathbf{r})\,\mathrm{d}V. \tag{5.2}$$

For an electrostatic situation, the value of U_{12} depends only on the distance between the two charges, r_{12}, where r_{12} is the magnitude of the relative separation vector

$$\mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2. \tag{5.3}$$

We write $\mathbf{r}_{12} = r_{12}\hat{\mathbf{r}}_{12}$ where $\hat{\mathbf{r}}_{12}$ is its direction.

By the usual Hamiltonian principle, the change of this system's energy, when expressed as a function of the separation of the charges, gives the force that acts on each charge. Thus the electromagnetic force, $\mathbf{F}_{12}$, due to charge q_2 on charge q_1 is the ratio of the energy change to the position change

$$\begin{aligned}\mathbf{F}_{12} &= -\lim_{\delta r_{12}\to 0}\frac{\delta U_{12}}{\delta r_{12}}\hat{\mathbf{r}}_{12}, \\ &= -\nabla_{12}U_{12}.\end{aligned} \tag{5.4}$$

There are some notational subtleties here, in that $\mathbf{F}_{12}$ is the force on q_1 due to q_2, U_{12} is the total energy in the electric field due to the system of $q_1 + q_2$, $\hat{\mathbf{r}}_{12}$ is the unit vector from q_1 to q_2. Finally, ∇_{12} denotes the change in U_{12} as function of variations in $\mathbf{r}_{12}$.

Explicitly, if we hold a small charged sphere q_1 stationary at $\mathbf{r}_1$, and move a second small charged sphere q_2 at $\mathbf{r}_2$ by a distance $\delta\mathbf{r}_2$, then $\delta\mathbf{r}_{12} = \delta\mathbf{r}_2$. Section 5.3 will evaluate the integral (5.2) for two charged spheres, and then differentiate it with respect to $\delta\mathbf{r}_2$ to obtain the usual Coulomb force law and consequently the Lorentz force law. Section 5.4 uses similar steps to calculate the force on a test charge inside a parallel plate capacitor. Some interesting pedagogical issues arise in these familiar situations.

In summary, this chapter shows that we may compute the force on a charge (or current) due to the field of another charge (or current) as the interaction of one field on another. The idea of fields acting on fields has also been considered by Williamson and van der Mark [11], who have used this principle to derive the origin of the exclusion principle. In terms of the underlying conceptual ideas, Coulomb's law was first expressed in terms of action at a distance.

$$\mathbf{F}_{12}^{\text{Coulomb}} = -\frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}}\hat{\mathbf{r}}_{12}. \tag{5.5}$$

The introduction of the field concept in the nineteenth century allowed his law to be replaced by a local interaction between the field due to one charge, and the charge of the other

$$\begin{aligned}\mathbf{F}_{12}^{\text{Coulomb, field}} &= \mathbf{E}_1 q_2, \\ \mathbf{F}_{21}^{\text{Coulomb, field}} &= \mathbf{E}_2 q_1,\end{aligned} \tag{5.6}$$

which satisfy Newton's Third Law, as $\mathbf{F}_{12}^{\text{Coulomb, field}} = -\mathbf{F}_{21}^{\text{Coulomb, field}}$.

In these expresssions one may first calculate the field due to one charge or the other, but not both, and then use the other charge in the formula. This choice can be a source of confusion to students new to the topic. Two questions arise for them in their classes, and for us in this chapter: "Why is the expression not symmetric?" and "Why don't I use both fields?"

We show that we may indeed obtain the same Coulombic force law by a symmetric expression only in the fields $\mathbf{E}_1$ and $\mathbf{E}_2$, rather than one field and the other charge, or both charges at once. In other words, eq(5.1) to eq(5.4) are equivalent to eqs(5.5 to 5.6). Once we have a field-field version of Coulombs' law, we may obtain the Lorentz force in the usual way, by a relativistic boost.

Our new approach will require a re-interpretation of the usual statements that electromagnetic fields do not interact with other electromagnetic fields in Maxwellian electromagnetism, but only with charges and currents (for an example see page 226 of reference [31]).

We observe that Coulomb's force law, eq(5.5), treats the two charges symmetrically, but involves action at a distance between charges. Action at a distance is contrary to the precepts of Lorentz relativity. The Lorentz force law, eq(5.6), involves the action of a field (due to one charge) on the other charge, and is thereby unsymmetrical. However it allows the use of a retarded field, one that corrects for the propagation time of the field, thereby removing the "instantaneous action at a distance" aspect of Coulomb's law. The present result is both symmetrical and expresses the force in terms of fields which can be evaluated in terms of retarded fields, and thus our result is essentially local.

In the next section we summarise some of the history of the ideas that led to Coulomb's law and to the concepts of fields.

5.2 The electromagnetic field

In the 1780's Coulomb found experimentally that there exists a force between two static charges separated in space and that the magnitude of this electrostatic force is directly proportional to the magnitude of each charge and inversely proportional to the square of the distance separating the two charges. Coulomb's law, as with Newton's law of gravity, is expressed in terms of action at a distance.

Action at a distance may be avoided by introducing the electric field. The electric field $\mathbf{E}_1$ at point $\mathbf{r}_2$ generated by the charge q_1 at $\mathbf{r}_1$ is equal to

$$\mathbf{E}_1 = \frac{q_1}{4\pi\epsilon_0 r_{12}^2}\hat{\mathbf{r}}_{12}. \tag{5.7}$$

We say the charge q_1 interacts with the electric field $\mathbf{E}_2$, or equivalently the field $\mathbf{E}_2$ acts on the charge q_1 to give the force law of eq(5.6) called the electrostatic Lorentz force law. Despite the name, it first appeared in a paper by Maxwell in 1861 [32]. Three years later, in 1864, Maxwell included this force law as one of his original eight electromagnetic equations [33].

To visualize the mechanics of the electromagnetic force between two bodies, in 1852 Faraday introduced *lines of force* [34]. When iron filings are spread over paper and brought near a bar magnet, the iron filings orient themselves end to end in lines from one pole of the magnet to the other. Faraday intepreted these lines as being the lines of force. Faraday also showed experimentally that these lines of force do not fit action at a distance models [34]. The lines of force were modified by Maxwell to *tubes of force.* This modification allowed Maxwell to make fluidic assumptions about the force and to derive a mathematical theory of electromagnetic fields. Maxwell believed these tubes of force propogated through the ether, creating a tension between bodies that was the electromagnetic force [33, 35].

The Maxwell–Lorentz electrostatic force law not only resolves the issue of action at a distance, but also includes the principle of superposition. The principle of superposition is part of the definition of a vector field, and is thus intrinsic to all of Maxwell's equations. As with Coulomb's law, the force experienced by one charge due to a static discrete distribution of other charges may be calculated using the principle of superposition, either by adding the force vectors or by adding the field vectors.

Lifting the restriction that the charges be stationary with respect to one another introduces magnetic fields. There is no law equivalent to Coulomb's law for magnetism and action at a distance is not an issue that arises. The magnetic force on a charge q is calculated using the magnetic Lorentz force law

$$\mathbf{F}_{12}^{\text{Magnetic}} = q_1(\mathbf{v}_1 \times \mathbf{B}_2). \tag{5.8}$$

where $\mathbf{B}_2$ is the magnetic field produced from charges q_2 moving at some velocities $\mathbf{v_2}$ with respect to the laboratory frame.

The magnetic field at a point $\mathbf{r}_1$ can be calculated using the Biot-Savart law

$$d\mathbf{B}_2(\mathbf{r}_1) = \frac{\mu_0}{4\pi}\,\frac{i_2 d\mathbf{s}_2 \times \mathbf{r}_{12}}{r_{12}^3}. \tag{5.9}$$

relating an infinitesimal magnetic field $d\mathbf{B}_2$ at $\mathbf{r}_1$ due to a infinitesimal current element $i_2 d\mathbf{s}_2$ at $\mathbf{r}_2$.

The general Lorentz force law, which incorporates both electric and magnetic fields, is the sum of forces (5.6) and (5.8)

$$\mathbf{F}_L = q_1(\mathbf{E}_2 + \mathbf{v}_1 \times \mathbf{B}_2). \tag{5.10}$$

This law may be alternatively derived from the electrostatic force (5.6) in a frame where the magnetic field is zero, by means of a Lorentz boost. This Lorentz force law provides an electrodynamic theory of charges where both the electric and magnetic fields are mediators for electromagnetic force and both fields act on the charge q_1.

A modern derivation of Maxwell's equations using the principles of relativity to uniquely characterise the electromagnetic interaction is provided in chapter 18 of Doughty [36]. The argument is, in outline:

After deducing the structure of special relativity, one can ask for the simplest non-trivial vector field A_μ such that $A_\mu A^\mu$ is a Lorentz scalar. The derivative (or 4-curl) of this field is a second rank anti-symmetric tensor called the electromagnetic (or Faraday) tensor and written $F^{\mu\nu}$. Equation (18.24) of Doughty is

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{5.11}$$

The electric and magnetic field components are contained in this tensor, $E^i = F^{0i}$ and $B^i = F^{jk}$.

Demanding that the field equations associated with the electromagnetic field $F^{\mu\nu}$ be covariant with respect to the full Poincaré group and under charge conjugation leads Doughty to the familiar covariant form of Maxwell's equations

$$\partial_\nu F^{\mu\nu} = \mu_0 J^\mu \quad \text{and} \quad \partial_\nu \tilde{F}^{\mu\nu} = 0, \tag{5.12}$$

where the source J^μ is another 4-vector, and $\tilde{F}^{\mu\nu}$ is the dual tensor of $F^{\mu\nu}$.

With the use of Noether's theorem, the energy momentum tensor $T^{\mu\nu}$ of the electromagnetic field $F^{\mu\nu}$ is derived to be

$$T^{\mu\nu} = \frac{1}{\mu_0}\left(F^{\mu\lambda}F^{\nu}_{\lambda} - \frac{1}{4}\eta^{\mu\nu}F_{\lambda\pi}F^{\lambda\pi}\right). \tag{5.13}$$

This is equation (18.49) of Doughty. The energy density is equal to the T^{00} component of the electromagnetic energy momentum tensor, equation (18.50)

$$T^{00} = \frac{1}{2}(\epsilon_0 \mathbf{E}^2 + \frac{1}{\mu_0}\mathbf{B}^2). \tag{5.14}$$

In summary, Doughty derives Maxwell's equations from the transformation laws of special relativity. Together with Noether's theorem, this gives the energy momentum tensor from which the energy density of the electromagnetic field can be obtained. We extend the work of Doughty by showing that Coulomb's law and more generally the Lorentz force law are derivable, using Hamilton's principle, from this energy density.

Considering now two independent electric (or magnetic) fields $\mathbf{E}_1, \mathbf{E}_2$, (or $\mathbf{B}_1, \mathbf{B}_2$) the total electric or magnetic field at a point in space is given by the principle of superposition of fields. That is, the total electric (magnetic) field is the sum of the individual electric (magnetic) fields at that point:

$$\mathbf{E}_{1,2} = \mathbf{E}_1 + \mathbf{E}_2, \tag{5.15}$$

$$\mathbf{B}_{1,2} = \mathbf{B}_1 + \mathbf{B}_2. \tag{5.16}$$

The total electromagnetic energy density is

$$u_{12} = \frac{\epsilon_0}{2}(\mathbf{E}_1 + \mathbf{E}_2)^2 + \frac{1}{2\mu_0}(\mathbf{B}_1 + \mathbf{B}_2)^2, \tag{5.17}$$

which may be written as

$$u_{12} = u_1 + u_2 + u_{1,2}, \tag{5.18}$$

where u_1 and u_2 are the energy densities of sources' electromagnetic fields, and we will call $u_{1,2}$ the *interaction energy density* of the two field configuration. Explicitly,

$$u_{1,2} \quad = \quad \epsilon_0 \mathbf{E}_1 \cdot \mathbf{E}_2 + \frac{1}{\mu_0}\mathbf{B}_1 \cdot \mathbf{B}_2. \tag{5.19}$$

The total electromagnetic energy is found by integrating the total energy density

$$U_{12} \quad = \quad \int_V u_{12} dv, \tag{5.20}$$

$$\quad = \quad \int_V (u_1 + u_2 + u_{1,2}) dv, \tag{5.21}$$

$$U_{1,2} \quad = \quad \int_V u_{1,2} dv. \tag{5.22}$$

In the situations of the next sections we need only find $U_{1,2}$ as U_1 and U_2, the integrals u_1 and u_2 over V, are constant.

5.3 Charge–charge interactions

In this section we will evaluate the integral (5.22) for two charges, and then differentiate it with respect to $\delta\mathbf{r}_2$ to obtain the usual Coulomb force law.

The electric fields associated with the two charges q_1 and q_2 are given by Gauss' law, one of Maxwell's equations

$$\mathbf{E}_1(\mathbf{r}) = \frac{q_1}{4\pi\epsilon_0 r_1^2}\hat{\mathbf{r}}_1, \qquad \mathbf{E}_2(\mathbf{r}) = \frac{q_2}{4\pi\epsilon_0 r_2^2}\hat{\mathbf{r}}_2, \tag{5.23}$$

where r_1 and r_2 are the distances from q_1 and q_2 respectively to a given point $\mathbf{r}$ in space where the electric fields are measured. The vectors $\hat{\mathbf{r}}_1$ and $\hat{\mathbf{r}}_2$ point radially outward from the charges.

A pictorial representation of the two charge system is given in Figure 5.1. In the cylindrical coordinates (x, z, ϕ) of the figure, q_1 is at $z = -\frac{1}{2}r_{12}$ and q_2 at $z = \frac{1}{2}r_{12}$. We consider a ring which has its centre on the z-axis. The angle θ_{12} between the two electric fields is the same for any point on this ring. This symmetry makes the integration easier.

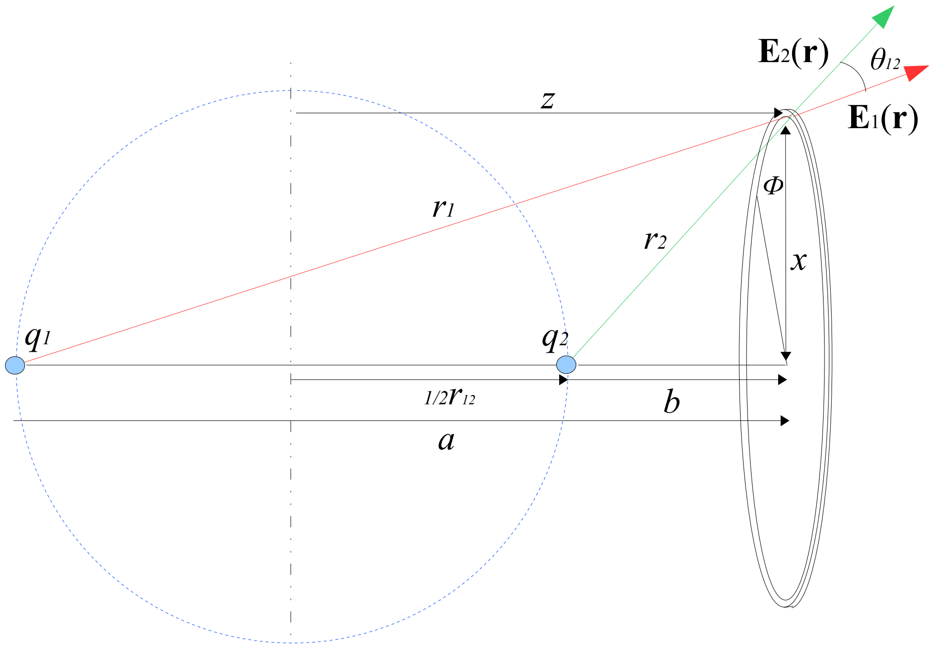

Figure 5.1: Two charges q_1 and q_2 separated in space by r_{12}. The origin for the cylindrical coordinates (x, z, ϕ) is midway between q_1 and q_2. The angle θ_{12} between the two electric fields is constant for any point on the ring shown.

The energy density of the system is

$$u_{12}(\mathbf{r}) = \frac{\epsilon_0}{2}(\mathbf{E}_1(\mathbf{r}) + \mathbf{E}_2(\mathbf{r}))^2, \tag{5.24}$$

$$= \frac{\epsilon_0}{2}E_1(\mathbf{r})^2 + \frac{\epsilon_0}{2}E_2(\mathbf{r})^2 + \epsilon_0 E_1(\mathbf{r})E_2(\mathbf{r})\cos\theta_{12}, \tag{5.25}$$

where θ_{12} is the angle from $\hat{\mathbf{r}}_1$ to $\hat{\mathbf{r}}_2$.

The integrals over all space of the first two terms, $u_1 = \frac{1}{2}\epsilon_0 E_1^2$ and $u_2 = \frac{1}{2}\epsilon_0 E_2^2$, are independent of the positions of q_1 and q_2, and so the u_1 and u_2 terms can be ignored in the calculations of the force. The interaction energy density term is

$$u_{1,2} = \epsilon_0 \mathbf{E}_1.\mathbf{E}_2, \tag{5.26}$$

$$= \epsilon_0 E_1 E_2 \cos\theta_{12}. \tag{5.27}$$

Substituting the expressions for E_1 and E_2 and using some trigonometry, see Figure 5.1, the interaction energy density may be rewritten as

$$u_{1,2} = \frac{q_1 q_2}{16\pi^2 \epsilon_0 r_1^3 r_2^3}(ab + x^2), \tag{5.28}$$

where $a = z + \frac{1}{2}r_{12}$ and $b = z - \frac{1}{2}r_{12}$. Substituting for r_1 and r_2 in terms of a, b and x puts the interaction energy density in the form

$$u_{1,2} = \frac{q_1 q_2}{16\pi^2 \epsilon_0} f(x, z), \tag{5.29}$$

where

$$f(x, z) = \frac{(ab + x^2)}{(a^2 + x^2)^{3/2}(b^2 + x^2)^{3/2}}. \tag{5.30}$$

To obtain $U_{1,2}$, the total interaction energy, we integrate with respect to angle ϕ, then by the radius of the ring x to get a disc, and thirdly integrate with respect to z. The integral with respect to ϕ is trivial, leaving us with the following expression for the total interaction energy

$$U_{1,2} = \frac{q_1 q_2}{8\pi\epsilon_0} \int_{z=-\infty}^{\infty} \int_{x=0}^{\infty} f(x, z)\, x\, dx\, dz. \tag{5.31}$$

The indefinite x-integral $\int f(x, z)\, x\, dx$ is easily evaluated,

$$\int f(x, z) dx = \frac{(ab + x^2)x}{(a^2 + x^2)^{3/2}(b^2 + x^2)^{3/2}} dx, \tag{5.32}$$

$$= \frac{x^2 - ab}{(a + b)^2(a^2 + x^2)^{1/2}(b^2 + x^2)^{1/2}}. \tag{5.33}$$

except at the special point $z = 0$, when $a + b = 0$, and for $z = \pm\frac{1}{2}r_{12}$. The values at these three points may be obtained by continuity considerations.

The $x = \infty$ limit gives $1/(a + b)^2$ but the $x = 0$ limit is more subtle giving $-ab/(a + b)^2|a||b|$. For z between q_1 and q_2, $ab/|a||b| = -1$, so the x-integral is zero. This is because the contribution to $\int f(x, z)\, x\, dx$ from x in the range from 0 to the surface of a sphere of radius $\frac{1}{2}r_{12}$ centred at the coordinate origin, cancels (for each value of z), the contribution from there to ∞. Within this sphere $\theta_{12} > \frac{1}{2}\pi$ and the interaction energy density is negative (assuming q_1 and q_2 are the same sign).

For $|z| > \frac{1}{2}r_{12}$ the x-integral gives

$$\int_{x=0}^{\infty} f(x, z)\, x\, dx = \frac{1}{2z^2}, \tag{5.34}$$

since $z = (a + b)/2$.

Integrating now with respect to z, the interaction energy for the region of space where $|z| < \frac{1}{2}r_{12}$, is composed of two equal parts (inside and outside the sphere defined above) that are of opposite sign, as remarked above. Furthermore it may be shown that these parts are also equal in magnitude to the integral for $z > \frac{1}{2}r_{12}$

$$U_{1,2}^{\text{right}} = \frac{q_1 q_2}{8\pi\epsilon_0}\int_{z=\frac{1}{2}r_{12}}^{\infty}\frac{dz}{z^2}, \tag{5.35}$$

$$= \frac{q_1 q_2}{8\pi\epsilon_0 r_{12}}. \tag{5.36}$$

We thus find the total interaction energy to be

$$U_{1,2} = \frac{q_1 q_2}{4\pi\epsilon_0 r_{12}}, \tag{5.37}$$

and therefore the force between the two charges is given by,

$$\mathbf{F}_{12} = -\frac{\partial}{\partial r_{12}}U_{1,2}, \tag{5.38}$$

$$= \frac{q_1 q_2}{4\pi\epsilon_0 r_{12}^2}, \tag{5.39}$$

which is Coulomb's law.

We leave it as an exercise for the reader to use eq(5.33) to show that the total interaction energy between the electric fields $\mathbf{E}_1$ and $\mathbf{E}_2$ due to charges q_1 and q_2 integrates to zero over a sphere centred on q_1 if q_2 is outside that sphere. This corresponds to the standard result that when computing the Coulomb force between a uniform spherical shell of total charge q_1 and a point charge q_2, the shell may be replaced by a point charge q_1 at the center of the shell.

5.4 A charge inside a parallel plate capacitor

We next use the same method to calculate the force on a test charge inside a parallel plate capacitor. We assume our parallel plate capacitor to be an ideal capacitor with infinitely large plates so that the electric field may be considered constant and the field lines parallel, and such that the test charge does not disturb this.

The standard approach to calculating the force on q_2 inside the capacitor is by using the Lorentz law $\mathbf{F}_{21} = q_2\mathbf{E}_1$ where $\mathbf{E}_1$ is the constant electric field generated by the charges on the capacitor plates at the position of q_2. The force on q_2 can instead be calculated using Coulomb's law by summing over the charges on the capacitor plates. The force on a charge inside the capacitor can thus be calculated without introducing an electric field.

The expression for the energy density of the total electric field is the same as in the previous section. The self energy of the capacitor plates and of the test charge may again be ignored as they are constant. The total interaction energy is found by integrating the interaction energy density over the space between the capacitor plates as $\mathbf{E}_1$ is zero outside.

A pictorial representation of the system is given in Figure (5.2).

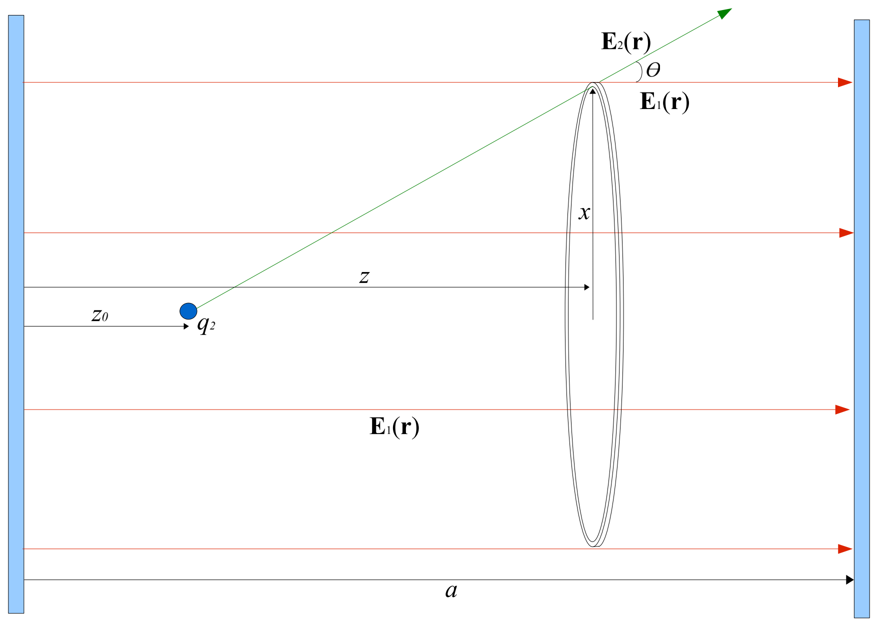

Figure 5.2: Charged particle inside an idealised capacitor. The origin for the cylindrical coordinates (x, z, ϕ) is chosen to be at the left hand plate.

Using the notation in Figure (5.2), and the expression (from Gauss' law) of the electric field of point charge q_2, the interaction energy density is

$$u_{1,2} = \frac{q_2 E_1}{4\pi} \frac{z - z_0}{\left((z - z_0)^2 + x^2\right)^{3/2}}. \tag{5.40}$$

Integrating as in the previous section, gives the total interaction energy as

$$U_{1,2} = \frac{q_2 E_1}{4\pi} \int_{z=0}^{a} \int_{x=0}^{\infty} \int_{\phi=0}^{2\pi} \frac{z - z_0}{\left((z - z_0)^2 + x^2\right)^{3/2}} x \, d\phi \, dx \, dz. \tag{5.41}$$

The integration with respect to ϕ is again trivial, and again care is needed with the limits for the x integration

$$\int_{x=0}^{\infty} \frac{z - z_0}{\left((z - z_0)^2 + x^2\right)^{3/2}} x \, dx = \left[\frac{(z - z_0) dz}{\left(x^2 + (z - z_0)^2\right)^{1/2}} \right]_0^{\infty}, \tag{5.42}$$

$$= \frac{(z - z_0)}{|z - z_0|}, \tag{5.43}$$

while the z integration is straightforward

$$U_{1,2} = \frac{q_2 E_1}{2} \int_{z=0}^{a} \frac{(z - z_0)}{|z - z_0|} dz, \quad (5.44)$$

$$= \frac{q_2 E_1}{2} \left(-\int_0^{z_0} dz + \int_{z_0}^{a} dz \right), \quad (5.45)$$

$$= \frac{q_2 E_1}{2} (a - 2z_0). \quad (5.46)$$

Note that the total interaction energy is zero if q_2 is half way between the plates, as then the interaction energy on the two sides cancel. Also, one might expect the force on q_2 would be mostly due to the interaction energy density nearby, but in this case of a idealised capacitor, the x integration gave a result independent of z.

The force on the charge inside the capacitor is given by minus the gradient of the interaction energy

$$\mathbf{F}_{21} = -\nabla_{z_0} U_{1,2}, \quad (5.47)$$

$$= -\frac{\partial U_{1,2}}{\partial z_0} \hat{\mathbf{z}}, \quad (5.48)$$

$$= q_2 \mathbf{E}_1, \quad (5.49)$$

which is the Lorentz force law for this situation.

5.5 Two parallel current-carrying wires

In the previous sections, Coulomb's law was derived as the gradient of the total interaction energy of a two charge system as was the force on a charge inside a capacitor. The charges in the system were stationary and hence only electric fields were considered. It was argued that the full Lorentz force law follows from a Lorentz boost. However it is instructive to calculate the force between two parallel current-carrying wires from the gradient of the interaction energy. A pictorial representation of the two wire system is given in Figure 5.3.

The standard way of calculating the force between two current carrying wires is to first calculate the magnetic field from one wire at any point in space using either the Biot-Savart law, eq(5.9), which in turn follows from Ampère's law, $\nabla \times \mathbf{H} = \mathbf{J}$, being a special case of one of Maxwell's equations. One finds that the magnetic field from a long wire has magnitude

$$B(\mathbf{r}) = \frac{\mu_0 i}{2\pi r}, \quad (5.50)$$

where r is the radial distance from the wire and i is the current flowing through the wire. The magnetic field $\mathbf{B}(\mathbf{r})$ circles the wire.

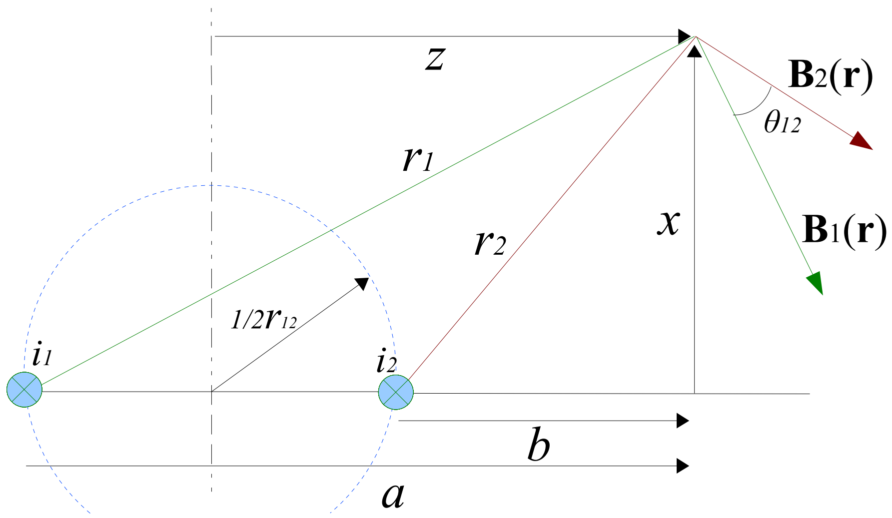

Figure 5.3: Two parallel wires, carrying currents i_1 and i_2 into the paper. The origin is at the midpoint between the wires and θ_{12} is the angle between the two magnetic fields at a point $\mathbf{r}$ in space.

The force between a section of length L of two parallel wires is given by the Lorentz force law. The field $\mathbf{B}_1(\mathbf{r})$ acts on all the charges q_2 moving at an average speed v_2 in the length L. If wire 2 is at position $\mathbf{r}_{12}$ relative to wire 1, we have

$$\mathbf{F}_{21} = q_2\mathbf{v}_2 \times \mathbf{B}_1(\mathbf{r}), \tag{5.51}$$

$$= \frac{\mu_0 i_1 i_2 \ell_2}{2\pi r_{12}}\hat{\mathbf{r}}_{12}. \tag{5.52}$$

The task is now to use the expression eq(5.50) to find the energy density of the magnetic field of two parallel wires, integrate that over all space, and differentiate the resulting expression to obtain the force law, eq(5.52), being a special case of the Lorentz force law.

The magnetic energy density from two parallel current carrying wires is

$$u_{12}(\mathbf{r}) = \frac{1}{2\mu_0}(\mathbf{B}_1(\mathbf{r}) + \mathbf{B}_2(\mathbf{r}))^2, \tag{5.53}$$

$$= \frac{1}{2\mu_0}B_1(\mathbf{r})^2, + \frac{1}{2\mu_0}B_2(\mathbf{r})^2 + \frac{1}{\mu_0}B_1(\mathbf{r})B_2(\mathbf{r})\cos\theta_{12}, \tag{5.54}$$

where θ_{12} is the angle between the two magnetic fields at $\mathbf{r}$. The first two terms represent the self energy of the currents and are independent of r_{12}, and so do not contribute to the force. The interaction energy density term is

$$u_{1,2}(\mathbf{r}) = \frac{1}{\mu_0} B_1(\mathbf{r}) B_2(\mathbf{r}) \cos\theta_{12}. \tag{5.55}$$

The geometry of this situation, see Figure (5.3), is very similiar to the two charge geometry, and we have

$$u_{1,2}(\mathbf{r}) = \frac{\mu_0 i_1 i_2}{4\pi^2} \frac{ab + x^2}{(a^2 + x^2)(b^2 + x^2)}. \tag{5.56}$$

where $a = z + \frac{1}{2}r$ and $b = z - \frac{1}{2}r$ as before.

The all-space integration is now over the rectangular coordinates x, y, z where $u_{1,2}(\mathbf{r})$ is independent of y

$$\begin{aligned} U_{1,2} &= \int_V u_{1,2}(\mathbf{r})\, dv, && (5.57)\\ &= \frac{\mu_0 i_1 i_2}{4\pi^2} \int_{z=-\infty}^{\infty} \int_{y=0}^{L} \int_{x=-\infty}^{\infty} \frac{ab + x^2}{(a^2 + x^2)(b^2 + x^2)}\, dx\, dy\, dz, && (5.58) \end{aligned}$$

so $\int dy$ contributes a factor of L.

The indefinite integral with respect to x is

$$\int \frac{ab + x^2}{(a^2 + x^2)(b^2 + x^2)} dx = 2\frac{\tan^{-1}(x/a) + \tan^{-1}(x/b)}{a + b}, \tag{5.59}$$

and, as before, we need to take some care with limits. When a and b are of the same sign, that is when $|z| > \frac{1}{2}r_{12}$ then the two $\tan^{-1}$ terms are both equal to $\pi/2$, and the x-integral gives $2\pi/(a + b) = \pi/z$. When a and b are of opposite signs, that is when z is between the wires, the $\tan^{-1}$ terms cancel. As with the two-charge configuration, these two terms have opposite signs because the angle between B_1 and B_2 is greater than $\pi/2$ for all points inside the circle centred midway between i_1 and i_2 that passes through i_1 and i_2, but less than $\pi/2$ outside.

The $1/z$ term gives a divergent result if a z-integration is performed. This divergence is a standard problem with this two-wire problem and arises from the non-physical model of the situation – the model assumes infinitely long parallel wires. These infinitely long wires lead to infinite self energies and an infinite interaction energy. The problem may be avoided by replacing the $z = \infty$ limits by a large but finite value, Z, and then finding ∇U_{12} before taking the limit. The interaction energy, given by eq(5.58), reduces to

$$U_{1,2} = \lim_{Z\to\infty} \frac{\mu_0 i_1 i_2 L}{4\pi} \left[\int_{z=-Z}^{-\frac{1}{2}r_{1,2}} \frac{1}{z} dz + \int_{z=\frac{1}{2}r_{1,2}}^{Z} \frac{1}{z} dz \right], \tag{5.60}$$

or

$$U_{1,2} = -\frac{\mu_0 i_1 i_2 L}{2\pi} \left[\ln(r_{12}/2) - \lim_{Z\to\infty} \ln Z \right]. \tag{5.61}$$

To calculate the force between the two current carrying wires, we differentiate the above equation with respect to r_{12}, the distance between the two wires.

$$-\frac{\partial}{\partial r_{12}} U_{1,2} = \frac{\mu_0 i_1 i_2 L}{2\pi} \frac{1}{r_{12}}, \tag{5.62}$$

and so the force per unit length between two wires due to the interaction energy is equal to

$$F_{21} = -\frac{\mu_0}{2\pi} \frac{i_1 i_2}{r_{12}}. \tag{5.63}$$

This result is in agreement with the result obtained using the Lorentz force law.

5.6 Summary

We have shown that the Lorentz force law follows from supplementing the energy density of the electromagnetic field by the Hamiltonian principle that force is related to the gradient of energy, $\mathbf{F} = -\nabla U$.

As an explicit demonstration of our claim we have evaluated the force between two charges at rest, a test charge in a capacitor, and the force between a pair of current-carrying wires. Boosting the result to a moving frame using the Lorentz transformation laws, which are the transformation laws for Maxwell's equations, leads to the full, relativistic Lorentz force law for the interactions between moving electric charges and electric currents.

Coulomb's law is written symmetrically in terms of the charges involved. It assumes action at a distance with no field to mediate the force, but action at a distance conflicts with Lorentz relativity. The development of the field concept, gave computational advantages over Coulomb's law in various situations, and led to the unification of magnetic fields into Maxwellian electromagnetism. Maxwellian electromagnetism is a single unified theory that includes predictions of electromagnetic wave propagation and allows for the introduction of retarded fields, removing problems with action at a distance.

The Lorentz force law has electric and magnetic fields as the mediators of electromagnetic force. The Lorentz force law states that the electromagnetic force on a (perhaps moving) test charge is dependent only on the electric and magnetic fields at the position of the test charge. What happens in the rest of space does not affect the force experienced by the test charge. The Lorentz force law is not directly symmetric with respect to the charge or charges that are the source of the field, and the charge that is being acted upon by the field.

The field interaction approach proposed here is different in several respects: It is symmetric with respect to the sources of the fields; it is Lorentz invariant and uses the retarded fields for all the fields contributing to the total field at each point in space; and the interaction energy (and hence the force experienced by each of the sources) is not determined simply by the electromagnetic field at the position of one of the charges but rather from the sum of the electromagnetic fields due to all charges throughout all space. While this last point may be seen as a disadvantage, we are able to derive the Lorentz force law from Maxwell's equations, together with Noether's theorem.

There are a number of implications of this approach, and there needs to be a re-interpretation of the electromagnetic field. In particular, electromagnetic fields are usually assumed not to interact with each other in Maxwellian electromagnetism. We have shown in this chapter that by laying aside this assumption, one may successfully calculate the electromagnetic forces between charged particles.

Chapter 6

Massless spin-one field equations in $C\ell(1,3)$

6.1 Introduction

Maxwell's equations are the cornerstone of electromagnetism. These equations, first discovered by James C. Maxwell and published in 1865 [33] provide a mathematical framework to describe most observed electromagnetic phenomena.

There are various notations in use to write down Maxwell's equations. Although originally these equations were written down by Maxwell in terms of twenty field variables, the most common way of writing Maxwell's equations today is in terms of the vector notation developed by Heaviside and Gibbs. The equations can also be written in terms of scalar and vector potentials or in terms of the Faraday tensor. In section 6.2 we write down Maxwell's equations in these different notations. Maxwell's equations can also be written down in the language of differential forms. The interested reader is referred to the text by Flanders [37].

Maxwell's equations can be derived and expressed in the Clifford algebra $C\ell(1,3)$. This is an already well established result, see for example [21]. A review of the formulation of the field equations within the Clifford algebra $C\ell(1,3)$ is provided in section 6.3. One advantage of the Clifford algebra formulation is that all four of Maxwell's equations are written as a single geometric equation $dF = -J$.

The second half of this chapter focuses on a set of equations named the *generalised* Maxwell equations in the literature [38, 39]. These equations are similar to Maxwell's equations but contain two extra fields. In section 6.4 we review how this set of equations is obtained as the massless limit of a spin one field of the Joos-Weinberg equation

$$\left(\gamma_{\{\mu\}} p^{[\mu]} - m^{2j} I\right) \psi(\mathbf{p}) = 0. \tag{6.1}$$

In section 6.5 we show how these generalised Maxwell equations may be derived in the Clifford algebra $C\ell(1,3)$. This derivation is very reminiscent of the derivation of the ordinary Maxwell equations, the difference being that we consider an eight component potential instead of a four component potential, and do not impose any Lorenz (or any other gauge) conditions.

Because the generalised Maxwell equations contain two extra fields, the energy conservation law is modified. In section 6.5 the energy conservation law for the generalised Maxwell equations is derived.

The first half of this chapter serves as an independent review of electromagnetsim in $C\ell(1,3)$. In the second half, the author's original contribution to the already existing work on the generalised Maxwell equations consists of showing that these equations can be derived within the Clifford algebra $C\ell(1,3)$ and that all the fields may be expressed in terms of an eight component potential, a mono-vector plus a tri-vector. To the author's awareness, the generalised energy conservation law of section 6.5 has not appeared in the literature and so this derivation is novel also.

6.2 Standard notations of Maxwell's equations

Maxwell originally wrote the equations that govern electromagnetism in terms of twenty field variables. Later, these equations were rewritten in the language of vector algebra by Heaviside [40] and Gibbs [41], which to date remains a widely used notation for Maxwell's equations.

6.2.1 Heaviside-Gibbs

In the Heaviside-Gibbs vector notation, the Maxwell equations are written as

$$\begin{aligned} \nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t} &= 0, && (6.2)\\ \nabla\times\mathbf{B}-\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} &= \mu_0\mathbf{J}, && (6.3)\\ \nabla\cdot\mathbf{E} &= \frac{\rho}{\epsilon_0}, && (6.4)\\ \nabla\cdot\mathbf{B} &= 0, && (6.5) \end{aligned}$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields respectively, ρ is the electric charge density and $\mathbf{J}$ is the current density. The first and fourth equations above are referred to as the homogeneous Maxwell equations, the second and third the inhomogeneous Maxwell equations. For the case where ρ and $\mathbf{J}$ are equal to zero we are left with the source free

Maxwell equations. One disadvantage of writing Maxwell's equations using the Heaviside-Gibbs vector notation is that when we consider Lorentz transformations, the fields do not transform simply. This is due to the fact that the 3-vector fields $\mathbf{E}$ and $\mathbf{B}$ are not part of relativistic 4-vectors, but elements of a rank 2 tensor.

6.2.2 Scalar and vector potential

Another common way to write Maxwell's equations is in terms of a scalar and a vector potential. Considering the Maxwell equation $\nabla \cdot \mathbf{B} = 0$ and the vector algebra identity $\nabla \cdot (\nabla \times \mathbf{A}) = 0$ for some vector $\mathbf{A}$, the magnetic field $\mathbf{B}$ can be written in terms of $\mathbf{A}$ as

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{6.6}$$

$\mathbf{A}$ is called the vector potential. Similarly, considering the Maxwell equation $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ together with the vector identity $\nabla \times (\nabla \phi) = 0$, where ϕ is some scalar function, the electric field $\mathbf{E}$ is expressed in terms of ϕ and $\mathbf{A}$ as

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi. \tag{6.7}$$

The two inhomogeneous Maxwell equations can now be written in terms of the scalar and vector potential as

$$\begin{aligned} \nabla^2 \phi + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) &= -\rho/\epsilon_0, && (6.8) \\ \nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla \left(\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} \right) &= -\mu_0 \mathbf{J}. && (6.9) \end{aligned}$$

Although (6.6) and (6.7) specify the magnetic and electric fields in terms of the potentials ϕ and $\mathbf{A}$, they do not specify them uniquely. Under the *gauge transformation*

$$\mathbf{A} \to \mathbf{A} - \nabla \chi, \qquad \phi \to \phi + \frac{\partial \chi}{\partial t}, \tag{6.10}$$

for some arbitrary scalar function χ, the expressions for the electric and magnetic fields remain unchanged. This degree of freedom can be used to choose χ in such a way that the potentials ϕ and $\mathbf{A}$ satisfy some particular *gauge condition*. One common choice is to choose χ in such a way that

$$\frac{1}{c^2}\frac{\partial \phi}{\partial t} + \nabla \cdot \mathbf{A} = 0. \tag{6.11}$$

This is called the Lorenz gauge condition.

Choosing the potentials so as to satisfy the Lorenz condition, the two inhomogeneous Maxwell equations written in terms of the potentials uncouple and leave the two inhomogeneous wave equations

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -\rho/\epsilon_0, \tag{6.12}$$

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J}. \tag{6.13}$$

The advantage of writing the fields and Maxwell's equations in terms of the scalar and vector potential is that the scalar and vector potential combine to form a relativistic 4-vector. This four vector behaves correctly under Lorentz transformations.

6.2.3 Four vectors and Faraday tensor

It is possible to write Maxwell's equations in a manifestly covariant way. Both the electric and magnetic field components can be written as components of a second rank tensor. Maxwell's equations are then written as two tensor equations. We will from now on set $c = \mu_0 = \epsilon_0 = 0$.

As mentioned earlier, the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ cannot be expressed as relativistic 4-vectors but the scalar and vector potentials ϕ and $\mathbf{A}$ can. We write $A_\mu = (\phi, \mathbf{A})$. The electric charge and current densities ρ and $\mathbf{J}$ also combine to give a 4-vector $J_\mu = (\rho, \mathbf{J})$. The right hand sides of equations (6.6) and (6.7) can be expressed as the components of the four dimensional curl of the 4-vector potential A

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \tag{6.14}$$

as may be easily verified. $F^{\mu\nu}$ is a second rank antisymmetric tensor called the *Faraday tensor*. In its matrix form, the Faraday tensor is written as

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}. \tag{6.15}$$

From the definition of $F^{\mu\nu}$, it is trivial to show that

$$\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = 0. \tag{6.16}$$

Setting μ, ν, λ equal to 1,2 and 3 respectively gives the equation

$$\partial^3 F^{12} + \partial^1 F^{23} + \partial^2 F^{31} = 0, \tag{6.17}$$

which can be rewritten as

$$\nabla \cdot \mathbf{B} \quad = \quad 0. \tag{6.18}$$

The other homogeneous equation may also be recovered by setting one of the indices in (6.16) equal to one. For example let $\lambda = 0$ and μ and ν equal to 1 and 2 respectively

$$\partial^0 F^{12} + \partial^1 F^{20} + \partial^2 F^{01} = 0. \tag{6.19}$$

This is the third component of the homogeneous Maxwell equation $\partial\mathbf{B}/\partial t + \nabla \times \mathbf{E} = 0$.

The inhomogeneous Maxwell equations are obtained by considering the action of the 4-derivative ∂_μ on the tensor and equating it to the 4-vector J^μ

$$\partial_\mu F^{\mu\nu} = J^\mu. \tag{6.20}$$

Consider $\nu = 0$. This gives

$$\partial_1 F^{10} + \partial_2 F^{20} + \partial_2 F^{30} \quad = \quad J^0, \tag{6.21}$$

which can be rewritten as

$$\nabla \cdot \mathbf{E} \quad = \quad \rho. \tag{6.22}$$

Similarly $\nu = 1$ gives

$$\partial_0 F^{01} + \partial_2 F^{21} + \partial_3 F^{31} \quad = \quad J^1. \tag{6.23}$$

In terms of the components of the electric and magnetic fields this is equal to

$$-\frac{\partial E^1}{\partial x_0} + \frac{\partial B^3}{\partial x_2} - \frac{\partial B^2}{\partial x_3} \quad = \quad J^1, \tag{6.24}$$

which is the first component of remaining inhomogeneous Maxwell equation, $-\partial\mathbf{E}/\partial t + \nabla \times \mathbf{B} = \mathbf{J}$.

Thus, all four of Maxwell's equations can be written more compactly in terms of two equations involving the Faraday tensor,

$$\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} \quad = \quad 0, \tag{6.25}$$

$$\partial_\mu F^{\mu\nu} \quad = \quad J^\nu. \tag{6.26}$$

Equation (6.25) is just Jacobi's identity.

Alternatively, the two homogeneous Maxwell equations may be obtained by introducing the dual Faraday tensor $\widetilde{F}^{\mu\nu}$ which is defined as

$$\widetilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}, \tag{6.27}$$

where $\epsilon^{\mu\nu\rho\sigma}$ is the four dimensional Levi-Civita symbol. In matrix form

$$\widetilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -B^1 & -B^2 & -B^3 \\ B^1 & 0 & E^3 & -E^2 \\ B^2 & -E^3 & 0 & E^1 \\ B^3 & E^2 & -E^1 & 0 \end{pmatrix}. \tag{6.28}$$

Because the Levi-Civita symbol is antisymmetric, the equation (6.16) may be written in terms of the dual Faraday tensor as $\partial_\mu \widetilde{F}^{\mu\nu} = 0$ and so Maxwell's equations may be more compactly written as

$$\begin{aligned} \partial_\mu \widetilde{F}^{\mu\nu} &= 0, \\ \partial_\mu F^{\mu\nu} &= J^\nu. \end{aligned} \tag{6.29}$$

The reader is reminded however that the introduction of the dual tensor is not required to write Maxwell's equations in terms of the Faraday tensor. The dual tensor was introduced merely for convenience.

6.3 $C\ell(1,3)$ formulation of Maxwell's equations

6.3.1 Maxwell's equations

In the standard Heaviside-Gibbs vector notation, the electric and magnetic fields **E** and **B** are both considered to be 3-vectors. As already mentioned, these field vectors are not parts of relativistic four vectors. Instead, the electric and magnetic fields are components of a rank 2 antisymmetric tensor. In $C\ell(1,3)$, the electric and magnetic fields are both represented as bi-vector quantities. Explicitly

$$\mathbf{E} = E_1 e_{01} + E_2 e_{02} + E_3 e_{03}, \tag{6.30}$$

$$\mathbf{B} = B_1 e_{23} + B_2 e_{31} + B_3 e_{12}. \tag{6.31}$$

These two fields combine into a single bi-vector F,

$$F = \mathbf{E} + \mathbf{B} = E_1 e_{01} + E_2 e_{02} + E_3 e_{03} + B_1 e_{23} + B_2 e_{31} + B_3 e_{12}. \tag{6.32}$$

Maxwell's equations are recovered from the single geometric equation

$$dF = -J, \tag{6.33}$$

where J is an appropriate source term. Consider the case of an electric current source

$$J = \rho e_0 + J_1 e_1 + J_2 e_2 + J_3 e_3, \tag{6.34}$$

where ρ is the electric charge density and $\mathbf{J} = J_1e_1 + J_2e_2 + J_3e_3$ is the electric current. Note that J is a 1-vector; if there are magnetic current sources as well then J has a nontrivial 3-vector component.

Explicitly, the equation $dF = -J$ becomes

$$\begin{aligned} dF = (e_0\partial_0 - e_1\partial_1 - e_2\partial_2 - e_3\partial_3)(E_1e_{01} + E_2e_{02} + E_3e_{03} \\ +B_1e_{23} + B_2e_{31} + B_3e_{12}) = -\rho e_0 - J_1e_1 - J_2e_2 - J_3e_3. \end{aligned} \tag{6.35}$$

Expanding the brackets and equating similar multivector components yields the set of equations:

$$(\partial_1E_1 + \partial_2E_2 + \partial_3E_3)e_0 = \rho e_0, \tag{6.36}$$

$$\begin{aligned} (-\partial_0E_1 + \partial_2B_3 - \partial_3B_2)e_1 + (-\partial_0E_2 + \partial_3B_1 - \partial_1B_3)e_2 + \\ (-\partial_0E_3 + \partial_1B_2 - \partial_2B_1)e_3 = J_1e_1 + J_2e_2 + J_3e_3, \end{aligned} \tag{6.37}$$

$$\begin{aligned} (\partial_0B_1 + \partial_2E_3 - \partial_3E_2)e_{023} + (\partial_0B_2 - \partial_1E_3 + \partial_3E_1)e_{031} + \\ (\partial_0B_3 + \partial_1E_2 - \partial_2E_1)e_{012} = 0, \end{aligned} \tag{6.38}$$

$$-(\partial_1B_1 + \partial_2B_2 + \partial_3B_3)e_{123} = 0. \tag{6.39}$$

Reading off the coefficients and regarding $\mathbf{E}$, $\mathbf{B}$ as 3-vectors with components (E_1, E_2, E_3), (B_1, B_2, B_3) respectively, we obtain Maxwell's equations:

$$\nabla\cdot\mathbf{E} = \rho, \qquad \nabla\times\mathbf{B} - \frac{\partial\mathbf{E}}{\partial t} = \mathbf{J}, \qquad \nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0, \qquad \nabla\cdot\mathbf{B} = 0. \tag{6.40}$$

6.3.2 Fields from potentials

In the Clifford algebra, the electric and magnetic fields can be defined in terms of a four vector potential. This potential is considered to be a mono-vector in $C\ell(1,3)$

$$A = \phi e_0 + A_1e_1 + A_2e_2 + A_3e_3. \tag{6.41}$$

The derivative of this potential is equal to

$$dA = d\cdot A + d\wedge A. \tag{6.42}$$

This product between two mono-vectors yields a scalar term L, and the six component bi-vector F with which we are already familiar

$$L = d\cdot A, \qquad F = -d\wedge A. \tag{6.43}$$

The six component bi-vector consists of three space-time components and three space component that are equal to the electric and magnetic field components respectively.

Consider the scalar term $L = d \cdot A$. Calculating this explicitly we get

$$d \cdot A = \frac{\partial \phi}{\partial t} + \nabla \cdot \mathbf{A}, \tag{6.44}$$

where $\mathbf{A}$ is space component of the potential A.

The potential A is not uniquely determined by the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$, but rather there is some freedom in choosing A. It is always possible to choose A in such a way that dA is a pure bi-vector[1]. To see this consider $A' = A + df$ for some scalar function f. Then

$$\begin{aligned} dA' &= dA + d^2 f, & (6.45) \\ &= L - F + d^2 f. & (6.46) \end{aligned}$$

So choosing $f : df = -L$ means that

$$dA' = -F, \tag{6.47}$$

a bi-vector. From (6.44) it is seen that in order for the scalar part of dA to vanish we must have $L = \partial\phi/\partial t + \nabla \cdot \mathbf{A} = 0$, which of course is the Lorenz condition. We then see that the Lorenz condition on the potential A is equivalent to the requirement that dA is a bi-vector.

By imposing the Lorenz condition $L = 0$, Maxwell's equations can be written in terms of a mono-vector potential A as

$$d^2 A = -dF = J. \tag{6.48}$$

6.3.3 Proca equations

The Proca equations, the field equations for massive spin one particles, like the Maxwell equations, have a very natural derivation in the Clifford algebra. The derivation is almost identical to the derivation of Maxwell's equations. Using the same notation, the Proca equations for a particle of mass m may be written as

$$\begin{aligned} dA &= -F, & (6.49) \\ dF &= m^2 A - J. & (6.50) \end{aligned}$$

[1]It should be noted that choosing a particular gauge condition for A may restrict the physics and it is possible that in doing so, some of the physics is lost. Whether this is the case or not is not clear to the author at present.

Note that in the massless limit, the above equations reduce to Maxwell's equations as expected.

The Clifford algebra derivation of the Proca equations illustrates two ideas: first, that the mass of a particle couples to the potential A to behave like an additional source term. The second is that for the Proca equations any gauge freedom there was for Maxwell's equations is lost. The Lorenz condition is now a consequence since A determines F uniquely and F determines A uniquely for the Proca equations. For Maxwell's equations, A determines F uniquely but F does not determine A uniquely, giving some freedom in choosing A.

6.4 Generalized Maxwell equations

Maxwell's equations are field equations for massless spin one objects. Weinberg [12] and Joos [42] have found field equations satisfied by arbitrary spin objects. For the case of spin one, one would expect to recover Maxwell's equations in the massless limit. This turns out not to be true in general and the most general solutions to the equations of Weinberg and Joos are a set of equations more general than Maxwell's equations, containing extra scalar fields. In this section we review the origin of this set of equations, referred to as the generalised Maxwell equations. We show that a particularly straightforward derivation of these equations exists in $C\ell(1,3)$ and that, like the ordinary Maxwell equations, they may be written as a single geometric equation in $C\ell(1,3)$. To the author's awareness this is a novel result that is not found in the presently available literature.

6.4.1 Field equations for arbitrary spin objects

Between 1964 and 1969, Weinberg produced a set of three papers [12,43,44] that deal with Feynman rules for arbitrary spin objects. In those papers he developed field equations satisfied by finite mass objects of arbitrary spin j. In the first paper, he considers a single $2(2j+1)$ component field

$$\psi(\mathbf{p}) = \begin{pmatrix} \phi(\mathbf{p}) \\ \chi(\mathbf{p}) \end{pmatrix} \tag{6.51}$$

where $\phi(\mathbf{p})$ and $\chi(\mathbf{p})$ are two $(2j+1)$ components fields of the $(j,0)$ and $(0,j)$ representations of the Lorentz group respectively. The field $\psi(\mathbf{p})$ transforms according to the $(j,0)\oplus(0,j)$ representation.

On top of satisfying the Klein-Gordon equation, $\psi(\mathbf{p})$ has enough components to ensure that it also satisfies some other homogeneous field equation. Both Weinberg [12] and Joos [42] found this equation to be

$$\left(\gamma_{\{\mu\}}p^{[\mu]} - m^{2j}I\right)\psi(\mathbf{p}) = 0 \tag{6.52}$$

where μ is a set of $2j$ Lorentz indices and $p^{[\mu]}$ is a product of $2j$ contravariant energy-momentum vectors. The equation holds for arbitrary spin and for $j = \frac{1}{2}$ one obtains the Dirac equation.

In the second paper [43], Weinberg presents field equations for massless arbitrary spin objects. Using the same procedure as in the first paper, it is found that in addition to the Klein-Gordon equation, the $(2j+1)$ component (not $2(2j+1)$ component because in the massless case, the equations uncouple[2]) objects satisfy the additional field equations

$$(\mathbf{S}\cdot\nabla)\left[\mathbf{S}\cdot\nabla - i(\partial/\partial t)\right]\psi^R(\mathbf{p}) = 0, \tag{6.53}$$

$$(\mathbf{S}\cdot\nabla)\left[\mathbf{S}\cdot\nabla + i(\partial/\partial t)\right]\psi^L(\mathbf{p}) = 0, \tag{6.54}$$

where $\mathbf{S}$ is the usual spin-j representation of angular momentum. For the spin one-half case, one obtains the standard Weyl equations.

For the spin one case we have (in the chiral representation)

$$S_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad S_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \tag{6.55}$$

$$S_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad I^{(3)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{6.56}$$

and so explicitly we have

$$\mathbf{S}\cdot\nabla = \begin{pmatrix} 0 & -\partial_z & \partial_y \\ \partial_z & 0 & -\partial_x \\ -\partial_y & \partial_x & 0 \end{pmatrix}. \tag{6.57}$$

Making the identification $\psi^R(\mathbf{p}) = (\mathbf{E} + i\mathbf{B})$ and $\psi^L(\mathbf{p}) = (\mathbf{E} - i\mathbf{B})$, we can then solve the two equations

$$\left[\mathbf{S}\cdot\nabla - i(\partial/\partial t)\right]\psi^R(\mathbf{p}) = 0, \tag{6.58}$$

$$\left[\mathbf{S}\cdot\nabla + i(\partial/\partial t)\right]\psi^L(\mathbf{p}) = 0, \tag{6.59}$$

[2] Note that in the Clifford algebra, the equations may not uncouple. Whether they do or not is not clear to the author at present.

to obtain two (Faraday and Ampere's law) of the Maxwell source free equations for left and right circularly polarized radiation

$$\nabla \times [\mathbf{E} + i\mathbf{B}] - i(\partial/\partial t)[\mathbf{E} + i\mathbf{B}] = 0, \tag{6.60}$$

$$\nabla \times [\mathbf{E} - i\mathbf{B}] + i(\partial/\partial t)[\mathbf{E} - i\mathbf{B}] = 0. \tag{6.61}$$

The remaining two source free Maxwell equations (electric and magnetic Gauss's law) can however not be obtained from first principles in this formalism. It was shown by Gersten [45] however that all of the source free Maxwell equations may be derived from first principles by decomposing the relativistic dispersion relation in a similar way to that which can be used to derive the Dirac equation, see for example chapter 2 of Ryder [46].

Although solving (6.58) and (6.59) gives two of Maxwell's equations, it is not the most general solution to equations (6.53) and (6.54). This is because the matrix $(\mathbf{S} \cdot \nabla)$ is not invertible.

6.4.2 Kinematic Acausality

It has been noted by Erst and Ahluwalia [47] that both the massive equations of Joos and Weinberg (6.52) and the massless equations of Weinberg for arbitrary spin particles (6.53)-(6.54) allow unphysical solutions. Apart from the correct dispersion relation $E = \pm|\mathbf{p}|$, one also obtains the wrong dispersion relation $E = 0$.

To see this explicitly, consider the dispersion relations of equations (6.58) and (6.59). We have

$$\det(\mathbf{S} \cdot \nabla \pm i(\partial/\partial t)) = \mp E(E^2 - \mathbf{p}^2) = 0, \tag{6.62}$$

which gives the dispersion relation

$$E = \pm|\mathbf{p}|, \qquad E = 0. \tag{6.63}$$

The fact that one obtains unphysical dispersion relations has been called *kinematic acausality* by Ernst and Ahluwalia [47]. They showed in the same paper that in the massless limit $m \to 0$, any acausal solutions for the dispersion relation satisfied by the $(j, 0) \oplus (0, j)$ covariant spinors associated with the Joos-Weinberg equation (6.52) can be made to disappear.

There is thus a discrepancy between the two sets of equations, one which was resolved by Ernst and Ahluwalia by noting that although the massless Weinberg equations are consistent with the finite mass Joos-Weinberg equations, they are not implied by them. There exists however a unique way of taking the massless limit that ensures all acausality vanishes.

The implications this has on the theory of electromagnetism has been noted among others by Dvoeglazov [38, 39]. Taking the massless limit of the spin one field equations in the $(1,0) \oplus (0,1)$ representation space of the Lorentz group does not give rise to the usual Maxwell equations used to describe electromagnetic phenomena but instead gives rise to a set of equation more general than the Maxwell equation containing extra scalar terms. This new set of equations has been termed the *generalised Maxwell equations* in the literature.

6.4.3 Generalised Maxwell equations

Gersten [45] derives Maxwell's equations starting from the Klein-Gordon equation for a massless spin one field and then following a procedure similar to that used to derive the Dirac equation of a spin one-half field. This approach leads him to the equation

$$(E^2 - \mathbf{p}^2)\mathbf{\Psi} = (EI^{(3)} - \mathbf{p}\cdot\mathbf{S})(EI^{(3)} + \mathbf{p}\cdot\mathbf{S})\mathbf{\Psi} - \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}(\mathbf{p}\cdot\mathbf{\Psi}) = 0, \qquad (6.64)$$

which is equation (9) in his paper[3]. Here $(E, \mathbf{p})$ is the energy-momentum four vector, with $\mathbf{p} = (p_x, p_y, p_z)$, $\mathbf{S}$ is as before and $\mathbf{\Psi}$ is a complex wave function. The solution set of this equation as found by Gersten is

$$(EI^{(3)} + \mathbf{p}\cdot\mathbf{S})\mathbf{\Psi} = 0, \qquad (6.65)$$

$$(\mathbf{p}\cdot\mathbf{\Psi}) = 0. \qquad (6.66)$$

Interpreting E in equation (6.65) as the definition of the Hamiltonian, and using the standard expressions in Schrodinger quantum mechanics, we obtain

$$i\hbar\frac{\partial\mathbf{\Psi}}{\partial t} = -\hbar\nabla\times\mathbf{\Psi}, \quad -i\hbar\nabla\cdot\mathbf{\Psi} = 0. \qquad (6.67)$$

Following Kramers [48] and writing the complex wave function as $\mathbf{\Psi} = \mathbf{E} - i\mathbf{B}$, these equations become

$$\nabla\times(\mathbf{E} - i\mathbf{B}) = -i\frac{\partial(\mathbf{E} - i\mathbf{B})}{\partial t}, \qquad (6.68)$$

$$\nabla\cdot(\mathbf{E} - i\mathbf{B}) = 0. \qquad (6.69)$$

Separating these equations into their real and imaginary parts leads to the source free Maxwell equations.

[3]for the sake of brevity the derivation of this equation is not discussed in detail here but can be found in the cited reference.

It was noted by Dvoeglazov [49] that equation (6.64) is also satisfied under the choice

$$(EI^{(3)} + \mathbf{p}\cdot\mathbf{S})\mathbf{\Psi} = \mathbf{p}\chi, \tag{6.70}$$

$$(\mathbf{p}\cdot\mathbf{\Psi}) = E\chi, \tag{6.71}$$

where χ is an arbitrary scalar field (possibly complex). We write $\chi = \chi_R + i\chi_I$.

This solution set suggested by Dvoeglazov gives not Maxwell's equations but instead leads to a set of equations more general which includes χ. These equations are

$$\nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = \nabla\chi_I, \tag{6.72}$$

$$\nabla\times\mathbf{B} - \frac{\partial\mathbf{E}}{\partial t} = \nabla\chi_R, \tag{6.73}$$

$$\nabla\cdot\mathbf{E} = -\frac{\partial\chi_R}{\partial t}, \tag{6.74}$$

$$\nabla\cdot\mathbf{B} = \frac{\partial\chi_I}{\partial t}. \tag{6.75}$$

We refer to equations (6.72)-(6.73) as the generalised Maxwell equations. The physical implications of the additional complex scalar field χ are discussed in the papers by Gersten and Dvoeglazov.

One can interpret the derivatives of this field χ to be the electric and magnetic sources and currents. Equating the derivatives of the real part of χ to the electric source and current yields

$$\rho = -\frac{\partial}{\partial t}\chi_R, \qquad \mathbf{J} = \nabla\chi_R. \tag{6.76}$$

As noted by Lee [50], this approach leads to the possibility of interpreting the electromagnetic field as a self-interacting non-Abelian gauge field with no magnetic monopoles. The same approach for the imaginary component of χ leads to a magnetic source and current. Making the assumption that there are no magnetic monopoles is then equivalent to demanding that χ is a real scalar field. A similar approach has been investigated by Leary [22], who shows that by not imposing the Lorenz condition, the derivative of the scalar term L, dL, can be interpreted as the source term J in the Clifford algebra formulation of Maxwell's equations.

6.5 $C\ell(1,3)$ formulation of the generalised Maxwell equations

6.5.1 The generalised Maxwell equations from Clifford algebra

In section 6.3 we considered a mono-vector potential A. By differentiating A (twice) and imposing the Lorenz condition $L = 0$, the Maxwell equations were obtained. In this section we generalise the potential to a mono-vector plus tri-vector. Furthermore we do not impose any gauge conditions on this potential. Differentiating this potential (twice) gives the generalised Maxwell equations (6.72)-(6.73) discussed in the previous section.

A general odd vector potential $P \in C\ell^{-}(1,3)$ can be written $P = \alpha + \beta\mathbf{e}$ where $\alpha = (\alpha_0, \boldsymbol{\alpha})$ and $\beta = (\beta_0, \boldsymbol{\beta})$ are both mono-vectors. Explicitly,

$$P = \alpha_0 e_0 + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \beta_0 e_0 e + \beta_1 e_1 e + \beta_2 e_2 e + \beta_3 e_3 e. \tag{6.77}$$

Next, define $G = dP$ which is an even vector $G \in C\ell^{+}(1,3)$ and write G as

$$G = f + A_1 e_{01} + A_2 e_{02} + A_3 e_{03} + D_1 e_{23} + D_2 e_{31} + D_3 e_{12} + ge, \tag{6.78}$$

where f, A_i, D_i, g are scalars.

Recall from the Clifford algebra formulation of Maxwell's equations that $dA = F$ was a bi-vector. The bi-vectors form a six dimensional subspace. In this case $G = dP$ is an even vector. The even vector of $C\ell(1,3)$ span an eight dimensional subspace, (in fact $C\ell^{+}(1,3)$ is a sub algebra of $C\ell(1,3)$).

Solving the equation

$$dG = -(J_e + J_m e), \tag{6.79}$$

leads to the following set of equations

$$\begin{aligned}
\frac{\partial f}{\partial t} - \nabla \cdot \mathbf{A} &= \rho_e, &(6.80)\\
\frac{\partial \mathbf{A}}{\partial t} - \nabla f - \nabla \times \mathbf{D} &= J_e, &(6.81)\\
\frac{\partial g}{\partial t} - \nabla \cdot \mathbf{D} &= \rho_m, &(6.82)\\
\frac{\partial \mathbf{D}}{\partial t} - \nabla g + \nabla \times \mathbf{A} &= J_m, &(6.83)
\end{aligned}$$

where $J_e = (\rho_e, \mathbf{J_e})$ and $J_m = (\rho_m, \mathbf{J_m})$ are electric and magnetic source terms respectively. If we now substitute $\mathbf{E} = \mathbf{A}$, $\mathbf{B} = \mathbf{D}$, $f = \chi_R$ and $g = \chi_I$, and ignore the source terms, then the generalized Maxwell equations (6.72)-(6.73) are recovered.

$f, \mathbf{A}, \mathbf{D}$ and g may be expressed in terms of the potential P by calculating dP. Consider $d\alpha$ and $d\beta e$ separately. We have

$$\begin{aligned}
d\alpha &= (\partial_0\alpha_0 + \partial_i\alpha_i) + (\partial_0\alpha_i + \partial_i\alpha_0)e_{0i} + (-\partial_i\alpha_j + \partial_j\alpha_i)e_{ij},\\
&= (\partial_0\alpha_0 + \nabla\cdot\boldsymbol{\alpha}) + (\partial_0\boldsymbol{\alpha} + \nabla\alpha_0)_i e_{0i} - (\nabla\times\boldsymbol{\alpha})_k e_{ij}, \qquad (6.84)
\end{aligned}$$

where $i, j, k = 1..3$ with $i \neq j \neq k$.

Making the standard identifications between the mono-vector component of the potential and the electric and magnetic fields

$$\begin{aligned}\mathbf{E}_\alpha &= -\frac{\partial \boldsymbol{\alpha}}{\partial t} - \nabla\alpha_0,\\ \mathbf{B}_\alpha &= \nabla\times\boldsymbol{\alpha},\end{aligned}$$

the bi-vector part of $d\alpha$ is $F_\alpha = E_i e_{0i} + B_i e_{jk}$ and Maxwell's equations can be written as $dF_\alpha = -J_e$.

Similarly, the tri-vector component of the potential gives

$$\begin{aligned}d\beta e &= (\partial_0\beta_0 + \partial_i\beta_i)e + (\partial_0\beta_i + \partial_i\beta_0)e_{0i}e + (-\partial_i\beta_j + \partial_j\beta_i)e_{ij}e,\\ &= (\partial_0\beta_0 + \partial_i\beta_i)e + (\partial_0\beta_i + \partial_i\beta_0)e_{jk} + (-\partial_i\beta_j + \partial_j\beta_i)e_{0k},\\ &= (\partial_0\beta_0 + \nabla\cdot\boldsymbol{\beta}) + (\partial_0\boldsymbol{\beta} + \nabla\beta_0)_i e_{jk} - (\nabla\times\boldsymbol{\beta})_i e_{0i}. \qquad (6.85)\end{aligned}$$

Making the identifications between the tri-vector component of the potential and the electric and magnetic fields

$$\begin{aligned}\mathbf{E}_\beta &= -\frac{\partial \boldsymbol{\beta}}{\partial t} - \nabla\beta_0,\\ \mathbf{B}_\beta &= \nabla\times\boldsymbol{\beta},\end{aligned}$$

the bi-vector part of $d\beta e$ is $F_\beta = B_i e_{0i} + E_i e_{jk}$ and Maxwell's equations can be written as $dF_\beta = -J_m$.

The mono-vector part of the potential, α, gives rise to a copy of Maxwell's equations with electric sources. The tri-vector part of the potential, βe, gives rise to a copy of Maxwell's equations with magnetic sources

$$\begin{aligned}dF_\alpha &= -J_e \quad \rightarrow \quad \text{electric charges and currents}, \qquad (6.86)\\ dF_\beta &= -J_m \quad \rightarrow \quad \text{magnetic charges and currents}. \qquad (6.87)\end{aligned}$$

Finally dP can be written as

$$\begin{aligned}dP &= d\alpha + d\beta e,\\ &= (\partial_0\alpha_0 + \partial_i\alpha_i) + (\partial_0\alpha_i + \partial_i\alpha_0 + \partial_j\beta_k - \partial_k\beta_j)e_{0i}\\ &\quad +(-\partial_i\alpha_j + \partial_j\alpha_i + \partial_0\beta_k - \partial_k\beta_0)e_{ij} + (\partial_0\beta_0 + \partial_i\beta_i)e,\\ &= (\frac{\partial\alpha_0}{\partial t} + \nabla\cdot\boldsymbol{\alpha}) + (\frac{\partial\boldsymbol{\alpha}}{\partial t} + \nabla\alpha_0 + \nabla\times\boldsymbol{\beta})_i e_{0i} + (-\nabla\times\boldsymbol{\alpha} + \frac{\partial\boldsymbol{\beta}}{\partial t} + \nabla\beta_0)_i e_{jk}\\ &\quad +(\frac{\partial\beta_0}{\partial t} + \nabla\cdot\boldsymbol{\beta})e. \qquad (6.88)\end{aligned}$$

This is equal to G as in (6.78) and so the fields $f, \mathbf{A} = \{A_1, A_2, A_3\}, \mathbf{D} = \{D_1, D_2, D_3\}$ and g are expressed in terms of the potential $P = \alpha + \beta e$ as

$$\begin{aligned}
f &= \frac{\partial \alpha_0}{\partial t} + \nabla \cdot \boldsymbol{\alpha}, && (6.89)\\
\mathbf{A} &= \frac{\partial \boldsymbol{\alpha}}{\partial t} + \nabla \alpha_0 + \nabla \times \boldsymbol{\beta}, && (6.90)\\
&= \mathbf{E}_\alpha - \mathbf{B}_\beta e, && (6.91)\\
\mathbf{D} &= -\nabla \times \boldsymbol{\alpha} + \frac{\partial \boldsymbol{\beta}}{\partial t} + \nabla \beta_0, && (6.92)\\
&= \mathbf{B}_\alpha + \mathbf{E}_\beta e, && (6.93)\\
g &= \frac{\partial \beta_0}{\partial t} + \nabla \cdot \boldsymbol{\beta}. && (6.94)
\end{aligned}$$

6.5.2 Energy conservation law

Associated with Maxwell's equations is an energy conservation law. In this section the energy conservation law for the generalised Maxwell equations is derived. This energy conservation law differs from the energy conservation law of Maxwell's equations. It contains extra terms involving the fields f and g.

The energy conservation law for Maxwell's equations is

$$\frac{\partial}{\partial t}\left(\frac{\mathbf{E}^2 + \mathbf{B}^2}{2}\right) + \nabla \cdot (\mathbf{E} \times \mathbf{B}) = 0. \tag{6.95}$$

Expanding (6.95) by using the vector identity $\nabla \cdot (\mathbf{X} \times \mathbf{Y}) = \mathbf{Y} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{B})$ gives

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{\mathbf{E}^2 + \mathbf{B}^2}{2}\right) + \nabla \cdot (\mathbf{E} \times \mathbf{B}) &= \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} + \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} + \mathbf{B} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{B}),\\
&= \mathbf{E} \cdot \left(\frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B}\right) + \mathbf{B} \cdot \left(\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E}\right), \quad (6.96)
\end{aligned}$$

which is equal to zero since the terms inside the parenthesis are two of Maxwell's equations. This shows that equation (6.95) holds for Maxwell's equations. For the generalised Maxwell equations however this equation would not hold because in addition to $\mathbf{E}$ and $\mathbf{B}$, f and g also enter into the energy conservation law.

For the generalised Maxwell equations, the equation equivalent to (6.96) is

$$\mathbf{E} \cdot \left(\frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B} - \nabla f\right) + \mathbf{B} \cdot \left(\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} - \nabla g\right) = 0, \tag{6.97}$$

where the Maxwell equations in (6.96) have been replaced by two of the generalised Maxwell equations. Comparison with (6.96) gives

$$\mathbf{E} \cdot \left(\frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B}\right) + \mathbf{B} \cdot \left(\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E}\right) = \mathbf{E} \cdot \nabla f + \mathbf{B} \cdot \nabla g. \tag{6.98}$$

From this equation the energy conservation law for the generalised Maxwell equations can be derived.

Note that $\mathbf{E}\cdot\nabla f$ and $\mathbf{B}\cdot\nabla g$ may be written as

$$\mathbf{E}\cdot\nabla f = \nabla\cdot(f\mathbf{E}) - f\nabla\cdot\mathbf{E}, \tag{6.99}$$

$$\mathbf{B}\cdot\nabla g = \nabla\cdot(g\mathbf{B}) - g\nabla\cdot\mathbf{B}. \tag{6.100}$$

Substituting these identities into (6.98) and using (6.95) gives

$$\frac{\partial}{\partial t}\left(\frac{\mathbf{E}^2+\mathbf{B}^2}{2}\right) + \nabla\cdot(\mathbf{E}\times\mathbf{B}) = \nabla\cdot(f\mathbf{E}) - f\nabla\cdot\mathbf{E} + \nabla\cdot(g\mathbf{B}) - g\nabla\cdot\mathbf{B}, \tag{6.101}$$

which may be rewritten in the form

$$\frac{\partial}{\partial t}\left(\frac{\mathbf{E}^2+\mathbf{B}^2}{2}\right) + f\nabla\cdot\mathbf{E} + g\nabla\cdot\mathbf{B} + \nabla\cdot(\mathbf{E}\times\mathbf{B} - f\mathbf{E} - g\mathbf{B}) = 0. \tag{6.102}$$

The first three terms can be combined into one time derivative $\frac{\partial}{\partial t}$ term by rewriting the two generalised Maxwell equations

$$\frac{\partial f}{\partial t} - \nabla\cdot\mathbf{E} = 0, \qquad \frac{\partial g}{\partial t} - \nabla\cdot\mathbf{B} = 0, \tag{6.103}$$

as

$$f\nabla\cdot\mathbf{E} = f\frac{\partial f}{\partial t}, \qquad g\nabla\cdot\mathbf{B} = g\frac{\partial g}{\partial t}. \tag{6.104}$$

Grouping together the terms involving a time derivative finally gives

$$\frac{\partial}{\partial t}\left(\frac{f^2+\mathbf{E}^2+\mathbf{B}^2+g^2}{2}\right) + \nabla\cdot(\mathbf{E}\times\mathbf{B} - f\mathbf{E} - g\mathbf{B}) = 0. \tag{6.105}$$

This is the energy conservation law for the generalised Maxwell equations.

6.6 Summary

In this chapter the Clifford algebra $C\ell(1,3)$ has been used to construct a theory of electromagnetism. In the first half of the chapter, we wrote Maxwell's equations using the vector notation of Heaviside and Gibbs. In this language, the electric and magnetic fields are expressed as vectors. This is not suitable in relativistic theories as the fields do not transform correctly because they do not form natural four vectors. In an explicitly covariant formulation of Maxwell's equations, the electric and magnetic fields are contained together inside the antisymmetric second rank Faraday tensor. Such an approach allows one to write Maxwell's equations as just two equations

$$\partial_\mu \widetilde{F}^{\mu\nu} = 0, \qquad \partial_\mu F^{\mu\nu} = J^\nu.$$

In $C\ell(1,3)$, Maxwell's equations are written as a single geometric equation

$$dF = -J,$$

where F is a bi-vector. Assuming the Lorenz condition ($L = 0$), F can be written as the derivative of a mono-vector potential A. Maxwell's equations are then written $d^2A = -dF = J$.

The Proca equations describing massive spin-1 particles may also be derived in $C\ell(1,3)$ in much the same way as Maxwell's equations, with the only addition being a mass term. This mass term couples to the potential A to behave like an additional source term. F is now uniquely defined in terms of A and vice versa, and so the gauge freedom of A is lost.

By considering the Klein-Gordon equation for a spin one field and following the same procedure as one does to find the Dirac equation for a spin one half field, one obtains not the Maxwell equations as would be expected but instead a more general set of equations called the generalised Maxwell equations. This set of equation, in addition to the electric and magnetic fields, contains two addition fields: one scalar field and one pseudoscalar field.

Starting with a more general potential consisting of a mono-vector plus a tri-vector, and not imposing any gauge conditions (such as the Lorenz condition), the generalised Maxwell equations can be derived in $C\ell(1,3)$. These equations, like the ordinary Maxwell equations, can be written as a single geometric equation.

The energy conservation law for the generalised Maxwell equations differs from the energy conservation law for Maxwell's equations due to the presence of two extra scalar fields.

Chapter 7

The stabilised Poincaré-Heisenberg algebra

7.1 Introduction

Many quantum cosmological[1] proposals come with a modification of the Heisenberg and Poincaré algebras. Confining ourselves to Lie algebraic modifications, we argue that modifications of the Heisenberg algebra inevitably leads to a loss of continuity or homogeneity of the underlying physical space. In order to establish this result, we first review how, within a quantum framework, the homogeneity and continuity of physical space leads inevitably to the Heisenberg algebra. We then review general arguments that hint toward algebraic modifications encountered in quantum cosmology proposals. Next, we argue that a natural extension of physical laws to the Planck scale can be obtained by a Lie algebraic modification of the Poincaré and Heisenberg algebras in such a way that the resulting algebra is *immune* to infinitesimal perturbations in its structure constants. This resulting algebra is commonly referred to as the stabilised Poincaré-Heisenberg algebra, or SPHA for short.

We establish in section 7.5 of this chapter that theories of the aforementioned class inevitably leads to a breakdown of the homogeneous and continuous nature of the underlying physical space. Furthermore, we show that any cosmologically derived quantum

[1]The literature cited in this chapter frequently uses the term "quantum gravity" in relation to the stabilised Poincaré-Heisenberg algebra (SPHA). The author's opinion is that this term is overused in the literature and in many cases is used unjustified. It is not clear to the author where and how gravity enters the SPHA, and the analysis presented here is not directly related to quantum gravity. For this reason we have chosen to use the suitably weaker phrase "quantum cosmology". The use of this term in relation to the SPHA is justified by the fact that the structure constants of this algebra reflect various scales and evolutionary epochs of the universe.

effects may carry a strong polarisation and spin dependence.

The proposed quantum relativistic (cosmological) kinematical algebra leads us to consider two additional length scales ℓ_P and ℓ_C as well as a dimensionless parameter β. We note that Amelino-Camelia [51] and Kowalski-Glikman and Smolin [52] were led by heuristic grounds to consider a similar path. The presence of β has been noted in [53–55] with differing emphasis. Chryssomalakos and Okon [54] show that it is always possible to gauge away this dimensionless constant by a suitable redefinition of the generators. The overall meaning of β seems to not be well understood in the literature other than that β explores other, equally stable algebras. The reader is referred to [54, 56–60] for additional discussion of these and related issues. We show that the parameter β is closely related to the geometry of the underlying physical space and, if nonzero, will radically affect some of the quantum relativistic notions.

7.2 Brief overview of Lie-algebraic stability

Physical theories are approximations to the natural world and the physical constants involved cannot be known without some degree of uncertainty due to the limitations of the experiments used to measure these constants. Properties of a model that are sensitive to small changes in the model, in particular changes in the values of the parameters, are unlikely to be observed. It can thus be reasoned that one should search for physical theories which do not change in a qualitative matter under a small change of the parameters. Such theories are said to be physically *stable* [54, 61].

The concept of the physical stability of a theory is given a mathematical meaning as follows. A mathematical structure is said to be mathematically stable for a class of deformations if any deformation in this class leads to an isomorphic algebraic structure. More precisely, a Lie algebra is said to be stable if small perturbations in its structure constants lead to isomorphic Lie algebras.

We do not present here a detailed discussion on the theory of Lie-algebraic deformations, because we make no use of such a theory in this thesis. As will be discussed in the next chapter, Clifford algebras have built into them the idea of stability. The purpose of this section is to review the reasoning that has led physicists to the stabilised Poincaré-Heisenberg algebra. For thorough and complete treatments of the subject, the reader is referred to Gerstenhaber [62], Nijenhuis and Richardson [61], and Chryssomalakos and Okon [54], who all discuss at length the mathematical theory of algebraic deformations omitted here.

As argued by the authors of [53, 54], the idea of mathematical stability provides insight into the validity of a physical theory or the need for a generalization of the theory. If a theory is not stable, one might choose to deform it until a stable theory is reached. Such a stable theory is likely to be a theory of wider validity compared to the original unstable theory.

Lie-algebraic deformation theory has been successful historically. Snyder [63] in 1947 showed that the assumption that spacetime be a continuum is not required for Lorentz invariance. Snyder's framework however leads to a lack of translational invariance. Later in the same year, Yang [64] showed this can be corrected if one allows for spacetime to be curved. In the same paper Yang presented the complete Lie algebra associated with the suggested changes.

Faddeev [65] and Mendes [53] argue that, in hindsight, stability considerations could have predicted the relativistic and quantum revolutions of the last century. When considering the Poisson and Galilean algebras, one finds that the algebraic structures are unstable. However, algebraic deformations take one from the Poisson algebra of classical mechanics to the Heisenberg algebra of quantum mechanics and from the Galilean algebra of Galilean relativity to the Poincaré algebra of special relativity. These algebras of quantum mechanics and special relativity, unlike their classical counterparts, can for most purposes be considered algebraically stable. The process of stabilising the Poisson and Galilean algebras via algebraic deformation introduces two deformation parameters. These parameters turn out to be the physical constants $\frac{1}{c^2}$ and $\hbar$ for the Galilean algebra and Poisson algebra respectively. Chryssomalakos and Okon [54] explain that for both the Galilean and Poisson algebra cases, the deformed algebras are isomorphic for all non-zero values of $\frac{1}{c^2}$ and $\hbar$. The values of these deformation parameters are determined by experiment.

More recently the question has arisen whether it is possible, via similar stability considerations as those that give rise to the algebras of quantum mechanics and special relativity, to find an algebraic signature of quantum cosmology. The question addressed by Ahluwalia [66] is: what Lie-algebraic structure is carried by freely falling frames at the interface of the gravitational/cosmological and quantum realms?

Chryssomalakos and Okon [54, 67] showed that by a suitable identification of the generators, triply special relativity proposed by Kowalski-Glikman and Smolin [52] as a potential candidate for a theory of quantum gravity can be brought to a linear form and that the resulting algebra is same as Yang's algebra [64].

The standard quantum relativistic kinematical algebra consists of the Poincaré algebra extended by the coordinates X_μ which have been promoted to generator status. This is essentially the direct product of the Poincaré and Heisenberg algebras. In the last decade

Mendes [53] concluded that the resultant Poincaré-Heisenberg algebra is not a stable Lie algebra. Mendes showed however that the algebra can be stabilised. This stabilisation requires two additional length scales: one in the extreme short distance range, the other on the cosmological scale. These length scales are denoted by ℓ_P and ℓ_c respectively (following the notation used in [66])[2]. The stabilised algebra is again Yang's 1947 algebra. This algebra is commonly referred to as the stabilised Poincaré-Heisenberg algebra (SPHA).

7.3 The geometry of the Heisenberg algebra

We begin by reviewing the fundamental connection between the homogeneity and continuity of physical space and the Heisenberg algebra. This will allow us to understand how modification to the Heisenberg algebra inevitably leads to changes in the geometry of the underlying space.

The reader is directed to an argument that is presented, for example, by Isham in Section 7.2.2 of [68]. There it is shown that, in the general quantum mechanical framework, and under the following two assumptions,

- physical space is homogeneous,
- any spatial distance r can be divided in to two equal parts, $r = r/2 + r/2$,

it necessarily follows that the operator x associated with position measurements along the x-axis, and the generator of displacements d_x along the x-direction, satisfy $[x, d_x] = i$. If one now requires consistency with the elementary wave mechanics of Heisenberg, one must identify d_x with $p_x/\hbar$ (p_x is the operator associated with momentum measurements along the x-direction). This gives, $[x, p_x] = i\hbar$. Without any additional assumptions, the argument easily generalises to yield the entire Heisenberg algebra

$$[x_j, p_k] = i\hbar\delta_{jk}, \tag{7.1}$$

$$[p_j, p_k] = 0, \tag{7.2}$$

$$[x_j, x_k] = 0, \tag{7.3}$$

where x_j, $j = 1, 2, 3$, are the position operators associated with the three coordinate axes, where the observer is assumed to be located at the origin of the coordinate system.

[2]in reference [66] these length scales are identified as $\ell_P = \sqrt{\hbar G/c^3}$ and $\ell_C = \sqrt{3c^4/8\pi G\rho_{vac}} = \sqrt{1/\Lambda}$ where ρ_{vac} and Λ are the vacuum energy density and cosmological constant respectively. The values of the two length scales is not important here. What is important is that two length scales exist, one in the extreme short distance scale, the other on the cosmological scale.

Thus it is evident that a quantum description of physical reality, with spatial *homogeneity* and *continuity*, inevitably leads to the Heisenberg algebra. It follows that modifications of this algebra necessarily induce a change in the underlying geometry of the space and either the homogeneity or the continuity (or both) of physical space is lost.

7.4 Beyond the Heisenberg and Poincaré algebras

From an algebraic point of view much of the success of modern physics can be traced back to the Poincaré and Heisenberg algebras. Had the latter algebra been discovered before the former, the conceptual formulation and evolution of theoretical physics would have been significantly different. For instance, it is a direct implication of Heisenberg's fundamental commutator $[x_i, p_j] = i\hbar\delta_{ij}$ (with $i, j = 1, 2, 3$), that *events* should be characterised not only by their spatiotemporal location x_μ, but also by the associated energy momentum p_μ; and that should be done in a manner consistent with the fundamental measurement uncertainties inherent in the formalism. The reader may wish to come back to these remarks in the context of equation (7.24) where it is shown that in a specific sense the physical space that underlies the conformal algebra does indeed combine the notions of spacetime and energy momentum. Furthermore, as will be seen from equation (7.26) and the subsequent remarks, this interplay becomes increasingly important in the conformal phase of the universe as the value of the parameter β increases between $\beta = 1$ and $\beta = \sqrt{2}$.

In the mentioned description the interplay of the general relativistic and quantum mechanical frameworks becomes inseparably bound. To see this, consider the well-known thought experiment to probe spacetime at spatial resolutions around the Planck length $\ell_P \stackrel{\mathrm{def}}{=} \sqrt{\hbar G/c^3}$. If one does that, one ends up creating a Planck mass $m_P \stackrel{\mathrm{def}}{=} \sqrt{\hbar c/G}$ black hole. This fleeting structure carries a temperature $T \approx 10^{30} K$ and evaporates in a thermal explosion in $\approx 10^{-40}$ seconds. This, incidentally, is a long time – about ten thousand fold the Planck time $\tau_P \stackrel{\mathrm{def}}{=} \sqrt{\hbar G/c^5}$. The formation and evaporation of the black hole places a fundamental limit on the resolution with which spacetime can be probed.

The authors of [69, 70] have argued that once gravitational effects associated with the quantum measurement process are accounted for, the Heisenberg algebra, and in particular the commutator $[x_j, p_k]$, must be modified. The role of gravity in the quantum measurement process was also emphasised by Penrose [71].

> *From the above discussion, we take it as suggestive that an operationally-defined view of physical space (or, its generalisation) shall inevitably ask for the length scale, ℓ_P to play an important role.*

From a dynamical point of view, as early as late 1800's, the symmetries of Maxwell's equations were already suggesting a merger of space and time into one physical entity, spacetime [72]. Algebraically, these symmetries are encoded in the Poincaré algebra. The emergent unification of space and time called for a new fundamental invariant, c, the speed of light (already contained in Maxwell's equations). From an empirical point of view, the Michelson-Morley experiment established the constancy of the speed of light for all inertial observers, and thus re-confirmed, in the Einsteinian framework, the implications of the Poincaré spacetime symmetries.

Concurrently, we note that while in classical statistical mechanics it is the volume that determines the number of accessible states and hence the entropy, the situation is dramatically different in a gravito-quantum mechanical setting. One example of this assertion may be found in the well-known Bekenstein-Hawking entropy result for a Schwarzschild black hole, $S_{BH} = (k/4)(A/\ell_P^2)$; where k is the Boltzmann constant, and A is the surface area of the sphere contained within the event horizon of the black hole. Thus quantum mechanical and gravitational realms conspire to suggest the holographic conjecture [73–75]. The underlying physics is perhaps two fold: (a) contributions from higher momenta in quantum fields to the number of accessible states is dramatically reduced because these are screened by the associated event horizons; and (b) the accessible states for a quantum system are severely influenced by the behaviour of the wave function at the boundary.

> *From this discussion, we take it as suggestive that in quantum cosmology/gravity the new operationally-defined view of physical space shall inevitably ask for a cosmological length scale, ℓ_C.*

Many cosmological models assume that the universe at some time in the past knew of no inertial frames of Einstein. This is because massive particles had not yet appeared on the scene. For such a scenario, the spacetime symmetries are encoded in the conformal algebra. So, whatever new operational view of spacetime emerges, we want it to also incorporate a process by which one evolves from the "conformal phase" of the universe to the present.

> *Algebraically, we take it to suggest that there must be a mechanism that describes how the present day Poincaré-algebraic description relates to the conformal-algebraic description of the universe in the past.*

In the conformal phase of the universe, where leptons and quarks were yet to acquire mass, the operationally-accessible symmetries are not Poincaré but conformal. This is so because to define rest frames, so essential for operationally establishing the Poincaré algebra, one needs massive particles. In the transition when massive particles come to exist, the local algebraic symmetries of general relativity suffer an operational change. Consequently, for a universe with conformal symmetry, a general relativistic description of physical reality might require modification.

7.5 A stable quantum relativistic algebra

A Lie algebra incorporating the three italicised items in the previous subsection has been studied previously. It was inspired by Faddeev's mathematical analysis of the quantum and relativistic revolutions of the last century [65] and was followed up by Vilela Mendes in his 1994 paper [53]. The uniqueness of the said algebra was then explored through a Lie-algebraic investigation of its stability by Chryssomalakos and Okon, in 2004 [54]. Some of the physical implications were subsequently explored in references [59, 66]. Its importance was further noted in *CERN Courier* [76].

However, its candidacy for the algebra underlying quantum cosmology has been difficult to assert. This is essentially due to a perplexing observation made in reference [54] regarding the interpretation of the operators associated with the spacetime events. In this section we overcome this interpretational hurdle and argue that it contains all the desired features for such an algebra.

To this end we first write down what has come to be known as the Stabilised Poincaré-Heisenberg Algebra (SPHA) and then proceed with the interpretational issues. The SPHA contains the Lorentz sector (we follow the widespread physics convention which takes the $\mathcal{J}_{\mu\nu}$ as dimensionless and $\mathcal{P}_\nu$ as dimensionful)

$$[\mathcal{J}_{\mu\nu}, \mathcal{J}_{\rho\sigma}] = i\left(\eta_{\nu\rho}\mathcal{J}_{\mu\sigma} + \eta_{\mu\sigma}\mathcal{J}_{\nu\rho} - \eta_{\mu\rho}\mathcal{J}_{\nu\sigma} - \eta_{\nu\sigma}\mathcal{J}_{\mu\rho}\right). \tag{7.4}$$

This sector remains unchanged (as is strongly suggested by the analysis presented in [77]), as does the commutator

$$[\mathcal{J}_{\mu\nu}, \mathcal{P}_\lambda] = i\left(\eta_{\nu\lambda}\mathcal{P}_\mu - \eta_{\mu\lambda}\mathcal{P}_\nu\right). \tag{7.5}$$

These are supplemented by the following modified sector

$$[\mathcal{J}_{\mu\nu}, \mathcal{X}_\lambda] = i\left(\eta_{\nu\lambda}\mathcal{X}_\mu - \eta_{\mu\lambda}\mathcal{X}_\nu\right), \tag{7.6}$$

$$[\mathcal{P}_\mu, \mathcal{P}_\nu] = iq\alpha_1\mathcal{J}_{\mu\nu}, \tag{7.7}$$

$$[\mathcal{X}_\mu, \mathcal{X}_\nu] = iq\alpha_2\mathcal{J}_{\mu\nu}, \tag{7.8}$$

$$[\mathcal{P}_\mu, \mathcal{X}_\nu] = iq\eta_{\mu\nu}\mathcal{I} + iq\alpha_3\,\mathcal{J}_{\mu\nu}, \tag{7.9}$$

$$[\mathcal{P}_\mu, \mathcal{I}] = i\alpha_1\mathcal{X}_\mu - i\alpha_3\mathcal{P}_\mu, \tag{7.10}$$

$$[\mathcal{X}_\mu, \mathcal{I}] = i\alpha_3\mathcal{X}_\mu - i\alpha_2\mathcal{P}_\mu, \tag{7.11}$$

$$[\mathcal{J}_{\mu\nu}, \mathcal{I}] = 0. \tag{7.12}$$

The metric $\eta_{\mu\nu}$ is taken to have the signature $(1, -1, -1, -1)$. The SPHA is stable, except for the instability surface defined by $\alpha_3^2 = \alpha_1\alpha_2$ (see Figure 7.1).

Away from the instability surface the SPHA is immune to infinitesimal perturbations in its structure constants. This distinguishes SPHA from many of the competing algebraic structures because, as already mentioned in the previous section, a physical theory based on such an algebra is likely to be free from "fine tuning" problems. This is essentially self evident because if an algebraic structure does not carry this immunity, one can hardly expect the physical theory based upon such an algebra to enjoy the opposite.

The SPHA involves three parameters $\alpha_1, \alpha_2, \alpha_3$. The c and $\hbar$ arise in the process of the Lie algebraic stabilisation that takes us from the Galilean relativity to Einsteinian relativity, and from classical mechanics to quantum mechanics. Their specific values are fixed by experiment. Similarly, $\alpha_1, \alpha_2, \alpha_3$ owe their origin to a similar stabilisation of the *combined* Poincaré and Heisenberg algebra.

The Lie algebraic procedure for obtaining SPHA does not determine $\alpha_1, \alpha_2, \alpha_3$. Dimensional and phenomenological considerations, along with the requirement that we obtain physically viable limits, suggest the following identifications [66]:[3]

$$\alpha_1 = \frac{\hbar}{\ell_C^2}, \tag{7.13}$$

where ℓ_C is of the order of the Hubble radius, and therefore it depends on the cosmic epoch. The introductory remarks, and existing data suggest that [54]

$$\alpha_2 = \frac{\ell_P^2}{\hbar} \tag{7.14}$$

In the limit $\ell_P \to 0, \ell_C \to \infty, \beta \to 0, \mathcal{I} \to I$, the identity operator, the SPHA splits into Heisenberg and Poincaré algebras. In that limit, $\mathcal{X}_\mu \to x_\mu, \mathcal{P}_\mu \to p_\mu, \mathcal{J}_{\mu\nu} \to J_{\mu\nu}$, and

[3] In making the identifications it is understood that these may be true up to a multiplicative factor of the order of unity.

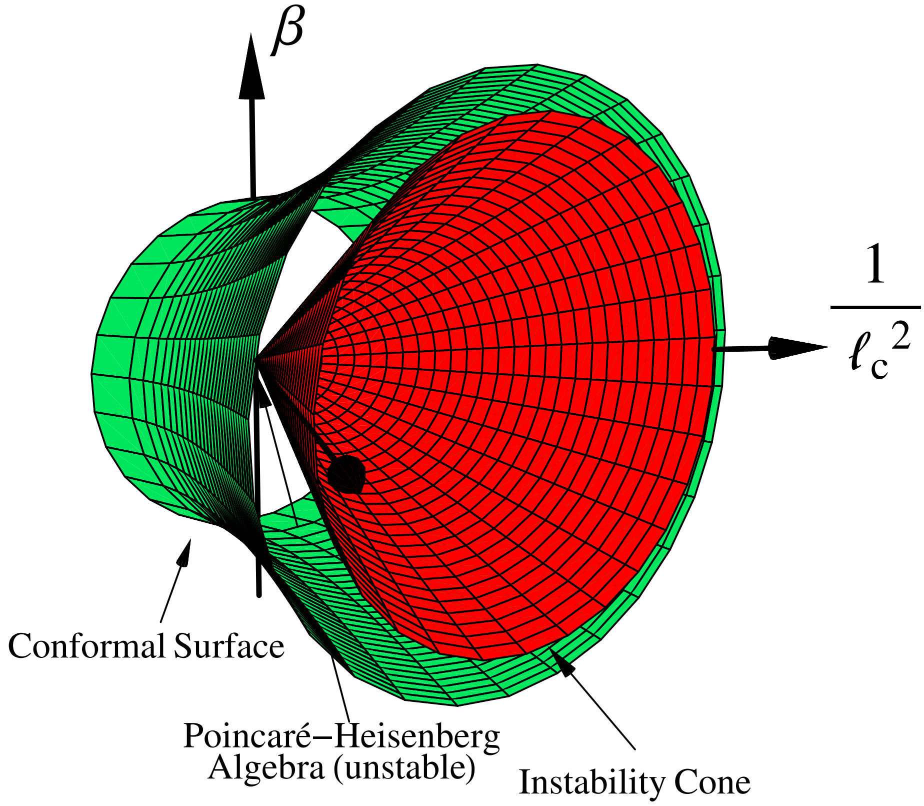

Figure 7.1: The unmarked arrow is the $\ell_P^2 (= \hbar\alpha_2)$ axis. The Poincaré-Heisenberg algebra corresponds to the origin of the parameters space, which coincides with the apex of the instability cone. In reference to Eq. (7.13), note that $\ell_C^2 = \hbar/\alpha_1$. Here, β is a dimensionless parameter that corresponds to a generalisation of the conformal algebra. The SPHA lives in the entire (ℓ_C, ℓ_P, β) space except for the surface of instability. The SPHA becomes conformal for all values of (ℓ_C, ℓ_P, β) that lie on the "conformal surface".

$\mathcal{I} \to I$. Thus $x_\mu, p_\mu, J_{\mu\nu}, I$ acquire their traditional meaning, while $\mathcal{X}_\mu, \mathcal{P}_\mu, \mathcal{J}_{\mu\nu}, \mathcal{I}$ are to be considered their generalisations. In particular x_μ should then be interpreted as the generator of energy-momentum translation. The latter parallels the canonical interpretation of p_μ as the generator of spacetime translation. This interpretation, we believe, removes the problematic interpretational aspects associated with $\mathcal{X}_\mu$ in the analysis of [54]. One might like to come back to these comments in light of section 9.3 where it is shown that the special conformal transformations act in an entirely similar way as translations but on different subspaces.

The identification of q with $\hbar$ is dictated by the demand that we recover the Heisenberg algebra. It also suggests that at the present state of the universe α_3 should not allow the second term in the right hand side of equation (7.9) to have a significant contribution. It will become apparent below that α_3 is intricately connected to the conformal algebraic limit of SPHA. With these identifications, and with α_3 renamed as the dimensionless parameter β, the SPHA takes the form

$$
\begin{aligned}
[\mathcal{J}_{\mu\nu}, \mathcal{J}_{\rho\sigma}] &= i\left(\eta_{\nu\rho}\mathcal{J}_{\mu\sigma} + \eta_{\mu\sigma}\mathcal{J}_{\nu\rho} - \eta_{\mu\rho}\mathcal{J}_{\nu\sigma} - \eta_{\nu\sigma}\mathcal{J}_{\mu\rho}\right), && (7.15)\\
[\mathcal{J}_{\mu\nu}, \mathcal{P}_{\lambda}] &= i\left(\eta_{\nu\lambda}\mathcal{P}_{\mu} - \eta_{\mu\lambda}\mathcal{P}_{\nu}\right), && (7.16)\\
[\mathcal{J}_{\mu\nu}, \mathcal{X}_{\lambda}] &= i\left(\eta_{\nu\lambda}\mathcal{X}_{\mu} - \eta_{\mu\lambda}\mathcal{X}_{\nu}\right), && (7.17)\\
[\mathcal{P}_{\mu}, \mathcal{P}_{\nu}] &= i\left(\hbar^2/\ell_C^2\right)\mathcal{J}_{\mu\nu}, && (7.18)\\
[\mathcal{X}_{\mu}, \mathcal{X}_{\nu}] &= i\ell_P^2\mathcal{J}_{\mu\nu}, && (7.19)\\
[\mathcal{P}_{\mu}, \mathcal{X}_{\nu}] &= i\hbar\eta_{\mu\nu}\mathcal{I} + i\hbar\beta\,\mathcal{J}_{\mu\nu}, && (7.20)\\
[\mathcal{P}_{\mu}, \mathcal{I}] &= i\left(\hbar/\ell_C^2\right)\mathcal{X}_{\mu} - i\beta\mathcal{P}_{\mu}, && (7.21)\\
[\mathcal{X}_{\mu}, \mathcal{I}] &= i\beta\mathcal{X}_{\mu} - i\left(\ell_P^2/\hbar\right)\mathcal{P}_{\mu}, && (7.22)\\
[\mathcal{J}_{\mu\nu}, \mathcal{I}] &= 0. && (7.23)
\end{aligned}
$$

As mentioned earlier, it is believed by many scientists that the universe at some time in the past knew of no inertial frames of Einstein, in which case the operationally accessible symmetries are not Poincaré but conformal. Hence, it should be encouraging if, in some limit, the SPHA reduced to the conformal algebra. This is indeed the case. It follows from a somewhat lengthy, though simple, exercise. Towards examining this question we introduce two new operators

$$\tilde{\mathcal{P}}_{\mu} = a\mathcal{P}_{\mu} + b\mathcal{X}_{\mu}, \quad \tilde{\mathcal{X}}_{\mu} = a'\mathcal{X}_{\mu} + b'\mathcal{P}_{\mu}, \tag{7.24}$$

and find that if the introduced parameters a, b, a', b' satisfy the the following conditions

$$a = \frac{\ell_P^2}{b'\hbar}, \quad b = \frac{1-\beta}{b'}, \quad a' = \frac{b'\hbar}{\ell_C^2(1-\beta)}, \tag{7.25}$$

and β^2 is restricted to the value $1 + (\ell_P^2/\ell_C^2)$, then SPHA written in terms of $\tilde{\mathcal{P}}_{\mu}$ and $\tilde{\mathcal{X}}_{\mu}$ satisfies the conformal algebra [78, Sec. 4.1].

Using these results, we can re-express $\tilde{\mathcal{P}}_{\mu}$ and $\tilde{\mathcal{X}}_{\mu}$ as follows

$$\tilde{\mathcal{P}}_{\mu} = a\left(\mathcal{P}_{\mu} + \frac{\hbar}{\ell_P^2}(1-\beta)\,\mathcal{X}_{\mu}\right), \quad \tilde{\mathcal{X}}_{\mu} = a'\left(\mathcal{X}_{\mu} + \frac{\ell_C^2}{\hbar}(1-\beta)\,\mathcal{P}_{\mu}\right), \tag{7.26}$$

with

$$\beta^2 = 1 + \frac{\ell_P^2}{\ell_C^2}. \tag{7.27}$$

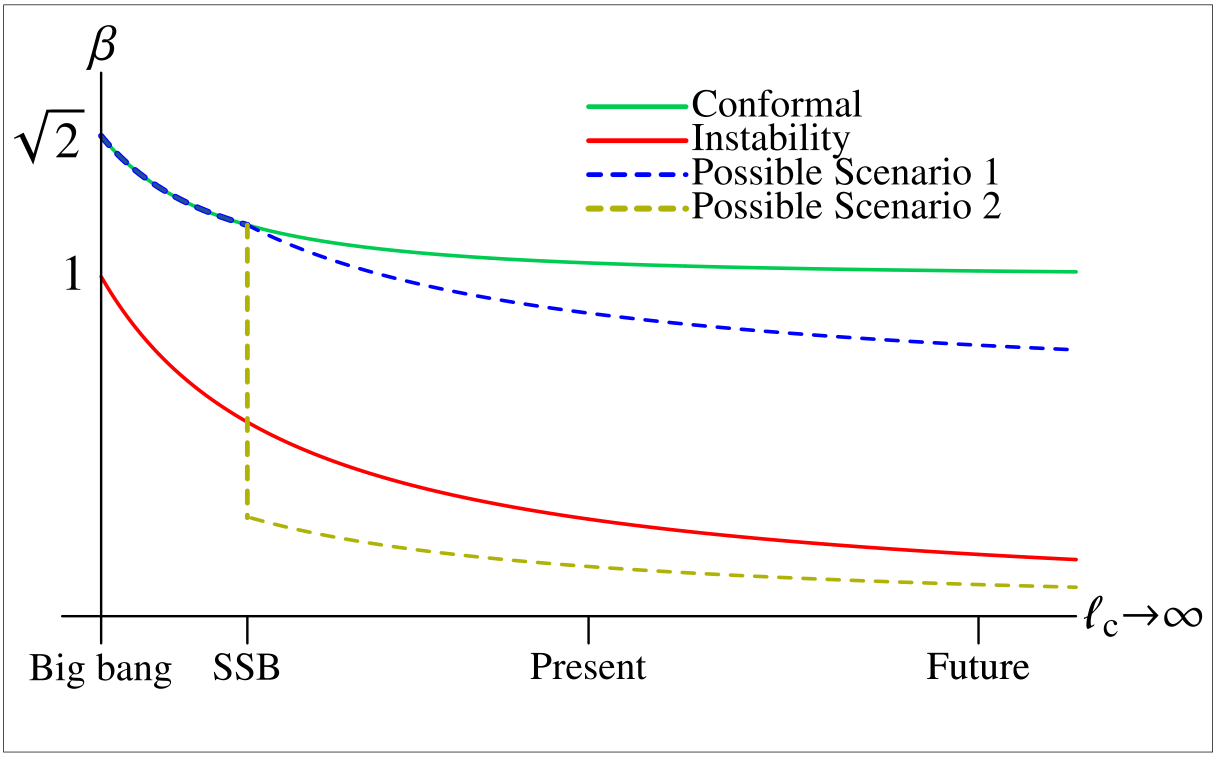

Figure 7.2: This figure is a cut, at $\ell_P = 1$ (with $\hbar$ set to unity), of Fig. 7.1 and it schematically shows the cosmic evolution along two possible scenarios. For this purpose, only $\beta \geq 0$ values have been taken. The $\beta < 0$ sector can easily be inferred from symmetry consideration. In one of the scenarios the conformal symmetry is lost without crossing the instability surface, while in the other it crosses that surface. This crossover, may be related to the mass-generating process of spontaneous symmetry breaking (SSB) of the standard model of high energy physics. We take $\ell_C \approx \ell_P$ to be the limit of what may be considered physically sensible.

In the limit $\ell_C \to \ell_P$ we have $\beta \to \pm\sqrt{2}$ (see, Figure 7.2). This results in a significant mixing of the $\mathcal{X}_\mu$ and $\mathcal{P}_\mu$ in the conformal algebraic description in terms of $\tilde{\mathcal{X}}_\mu$ and $\tilde{\mathcal{P}}_\mu$.

In contrast, hypothetically, had we been on the conformal surface at present then taking $\ell_C \gg \ell_P$ makes $\beta \to \pm 1$. Consequently, for $\beta \to +1$, $\tilde{\mathcal{P}}_\mu$ becomes identical to $\mathcal{P}_\mu$ up to a multiplicative scale factor a. Similarly, $\tilde{\mathcal{X}}_\mu$ becomes identical to $\mathcal{X}_\mu$ up to a multiplicative scale factor a'. As is evident from Eq. (7.25), the multiplicative scale factors a and a' are constrained by the relation $aa' = \ell_P^2/(\ell_C^2(1-\beta))$. We expect that similar modifications to spacetime symmetries would occur if we were to explore it at Planckian energies in the present epoch. For $\beta \to -1$ ($\ell_C \gg \ell_P$), one again obtains significant mixing of the $\mathcal{X}_\mu$ and $\mathcal{P}_\mu$.

By containing ℓ_P and ℓ_C, the SPHA unifies the extreme microscopic with the extreme macroscopic, i.e., the cosmological. It also allows for the existence of conformal symmetry under certain conditions. A significant departure from the Heisenberg algebra, when for example ℓ_C approaches ℓ_P, leads to a loss of continuity or homogeneity in the underlying physical space and the quantum fields that it supports. The latter is an unavoidable consequence of the discussion presented in section 2.[4]

Some of the authors of [5] have argued that it is the homogeneity of space that is lost and that the induced inhomogeneities may serve as seeds for structure formation. The author of this thesis challenges this claim, suggesting that the physical space is homogeneous but not continuous. In manuscript [9], the homogeneity and isotropy of space is used to derive the Clifford algebra $C\ell(1,3)$ from the underlying geometry of space. There is no assumption that the underlying space is continuous. In the next chapter, we show that the non-scalar basis elements of this Clifford algebra precisely generate the SPHA under the action of the Lie bracket. It seems therefore reasonable to conclude that the physical space underlying the SPHA should therefore be homogeneous. It follows that it cannot also be continuous. We note that we are defining both position and momentum in a single space (and algebra). A possible way to avoid the issue of homogeneity and continuity of space is by considering multiple homogeneous, continuous spaces. In chapter 9 we consider position and momentum in separate spaces to describe the conformal algebra in terms of two copies of the algebra $C\ell(1,3)$.

7.6 Polarisation and spin dependence of cosmologically derived quantum effects

An examination of the SPHA presented in equations (7.15-7.23) reveals a strong $\mathcal{J}_{\mu\nu}$ dependence of the modifications to the Heisenberg algebra. Physically, this translates to the following representative implications

- Any induced changes to the geometry of the physical space are dependent on spin and polarisation of the fields for which these are calculated.
- The operationally-inferred commutativity/non-commutativity of the physical space depends on the spin and polarisation of the probing particle.
- The previous observation implies that a violation of the equivalence principle is inherent in a SPHA based quantum cosmology.

[4]Any one of the other suggestions in quantum cosmology that modify the Heisenberg algebra (see, e.g., references [79–89]) carry similar implications for continuity, homogeneity and isotropy of the physical space.

- Since the Heisenberg algebra uniquely determines the nature of the wave particle duality [88, 89] (including the de Broglie result "$\lambda = h/p$"), it would undergo spin and polarisation dependent changes in a quantum cosmological theory based on SPHA.

All these results carry over to any theory of quantum cosmology or quantum gravity that modifies the Heisenberg algebra with a $\mathcal{J}_{\mu\nu}$ dependence.

7.7 Summary

In this chapter we have motivated the SPHA as a candidate for the kinematical algebra which may underlie a physically viable and consistent theory of quantum cosmology. Besides yielding an algebraic unification of the extreme microscopic and cosmological scales, it generalises the notion of conformal symmetry.

The modifications to the Heisenberg algebra at the present epoch of the universe are negligibly small; but when ℓ_C and ℓ_P are of the same order, the modifications are significant resulting in the breakdown of the continuous and homogeneous nature of physical space that underlies the Heisenberg algebra. It was argued by some of the authors of [5] that this space must be inhomogeneous and that such inhomogeneities could serve as an important ingredient for structure formation [90]. In light of the next chapter, we argue that the physical space is more likely to be homogeneous but not continuous.

Furthermore, in this class of theories one must expect a strong polarisation and spin dependence of various cosmologically derived quantum effects.

There is a limit in which SPHA reduces to the conformal algebra. This limit describes how the present day Poincaré-algebraic description relates to the conformal-algebraic description of the universe in the past. It was shown that the dimensionless parameter β is closely related to the geometry of the underlying physical space. The physical space that underlies the conformal algebra combines the notions of spacetime and energy momentum through the mixing of $\mathcal{X}_\mu$ and $\mathcal{P}_\mu$. The extent of this interplay is governed by the value of β.

Chapter 8

Clifford algebraic representations of the SPHA

8.1 Introduction

The last few decades has seen a search for a theory which successfully unifies physics at the quantum scale with physics at the cosmological. Part of this search has been to find an algebraic signature of quantum cosmology. Although many proposals for a theory of quantum cosmology and quantum gravity have been pursued, to date no satisfactory theory exists. We are however now entering an age where for the first time, experiments are sensitive enough to measure cosmologically induced quantum effects. Results from experiments such as CERN's LHC may falsify some of the theories proposed to date.

Among the proposals considered are string theory, loop quantum gravity, doubly and triply special relativity, non commutative geometry and Lie algebraic deformations. This list is certainly not complete. These various approaches are not mutually exclusive to one another but in many cases there is a significant overlap of ideas, especially with regards to the nature of the underlying physical space. For example, non-commutative spacetime is an element common to many proposed theories. One candidate for an algebraic signature is the kinematical algebra proposed in the previous chapter. Indeed the SPHA has received the attention of a few authors in recent years [5, 54, 66].

In this chapter we show that the Clifford algebra $C\ell(1,3)$ is consistent with an SPHA-based approach to quantum cosmology. Specifically, the commutation relations of the SPHA will be shown to arise from the non-scalar basis elements of $C\ell(1,3)$ under the action of the Lie bracket $[x,y] = xy - yx$.

It is hoped by the authors of [4] that this result is encouraging for both the community of people working in the field of Clifford algebras, and the community of people working with Lie algebraic deformations of the Heisenberg and Poincaré algebras. For the Clifford algebra community this is because there exists a well developed mathematical formalism to deal with Lie algebraic types of deformations, and for the second community because Clifford algebras are firmly linked to the underlying geometry of space. This was shown in chapter 2 and also [9].

This author believes that the Clifford algebra $C\ell(1,3)$ may aid in determining the nature of the underlying physical space of an SPHA-based theory of quantum cosmology. For example

- The spacetime of $C\ell(1,3)$ is non-commutative. The coordinates of space and time are non-commuting quantities, in agreement with the algebra of Yang [64] and non-commutative geometries considered by others, for example Connes [91].
- The spacetime of $C\ell(1,3)$ is homogeneous. This follows from chapter 2 where we showed that the homogeneity and isotropy of spacetime were required to derive the Clifford algebra.
- The algebra $C\ell(1,3)$ suggests indirectly that an SPHA spacetime is not continuous. This follows from the previous point and arguments presented in the previous chapter about modifications to the Heisenberg algebra.
- The algebraic stability of $C\ell(1,3)$ is assured.

This last point may require some justification. The Clifford algebra path to the SPHA avoids the traditional stability considerations and Lie-algebraic modifications. This is so because a Clifford algebra is determined to within an isomorphism by its metric which is simply a symmetric matrix with eigenvalues which are either positive or negative. As long as these are non-zero, a (small) perturbation will leave this signature unchanged, and hence stability follows. The Clifford algebra approach is conceptually easier as well as more straightforward. It avoids some of the complicated (at least in the eyes of the author) mathematics associated with Lie-algebraic deformations and stability.

We will show later in this chapter that the Clifford algebra path suggests that in the quantum relativistic realm, events should be characterised by both their spacetime location x_μ and their associated energy-momentum p_μ, not by their spacetime location alone. The Clifford algebra therefore combines the notions of spacetime and energy momentum. This is in agreement with the discussions in the previous chapter.

In the previous chapter it was shown that in the conformal algebraic limit of the SPHA, a mixing of the $\mathcal{X}_\mu$ and $\mathcal{P}_\mu$ generators takes place. The extent of the mixing is determined by the value of the parameter β. We show in section 8.3 that the parameter

$\alpha_3 = \beta$ induces a mixing of the $\mathcal{X}_\mu$ and $\mathcal{P}_\mu$ not only in the conformal algebraic limit but in the SPHA in general. We show that an expression for β can be found in terms of the two length scales ℓ_P, ℓ_C and an angle parameter φ. This result may contribute to further understanding how β affects various quantum relativistic notions.

8.2 Restricted Clifford algebraic representations

In this section we show that the SPHA can be generated from the non-scalar basis elements of the Clifford algebra $C\ell(1,3)$ under the action of the Lie bracket. In particular we find how the fifteen generators of the SPHA can be represented by these non-scalar elements. It seems a natural choice to choose the six bi-vectors to represent the six generators of Lorentz transformations $J_{\mu\nu}$ and the pseudoscalar to represent the generator iM. This leaves the mono-vectors and tri-vectors to represent the generators of momentum translation X_μ and the generators of translation P_μ.

We proceed by defining the generators

$$\begin{aligned} X_\mu &= \frac{1}{2}\sqrt{-q\alpha_2}e_\mu, && (8.1)\\ P_\mu &= \frac{1}{2}\sqrt{-q\alpha_1}ee_\mu, && (8.2)\\ iJ_{\mu\nu} &= \frac{-1}{2}e_{\mu\nu}, && (8.3)\\ iM &= \frac{-1}{2}\sqrt{\alpha_1\alpha_2}e. && (8.4) \end{aligned}$$

These generate the SPHA under the action of the Lie bracket with the restriction that the dimensionless parameter $\alpha_3 = \beta$ is equal to zero.

Explicitly, we get

$$\begin{aligned} [iJ_{\mu\nu}, iJ_{\rho\sigma}] &= -(\eta_{\nu\rho}iJ_{\mu\sigma} + \eta_{\mu\sigma}iJ_{\nu\rho} - \eta_{\mu\rho}iJ_{\nu\sigma} - \eta_{\nu\sigma}iJ_{\mu\rho}), && (8.5)\\ [iJ_{\mu\nu}, P_\lambda] &= -(\eta_{\nu\lambda}P_\mu - \eta_{\mu\lambda}P_\nu), && (8.6)\\ [iJ_{\mu\nu}, X_\lambda] &= -(\eta_{\nu\lambda}X_\mu - \eta_{\mu\lambda}X_\nu), && (8.7)\\ [P_\mu, P_\nu] &= q\alpha_1 iJ_{\mu\nu}, && (8.8)\\ [X_\mu, X_\nu] &= q\alpha_2 iJ_{\mu\nu}, && (8.9)\\ [P_\mu, X_\nu] &= q\eta_{\mu\nu}iM + q\alpha_3 iJ_{\mu\nu}, && (8.10)\\ [P_\mu, iM] &= -\alpha_1 X_\mu + \alpha_3 P_\mu, && (8.11)\\ [X_\mu, iM] &= -\alpha_3 X_\mu + \alpha_2 P_\mu, && (8.12)\\ [iJ_{\mu\nu}, iM] &= 0, && (8.13) \end{aligned}$$

where $\alpha_3 = 0$ and additional i's have been introduced for convenience.

We have here chosen to use $X_\mu,\ P_\mu,\ iJ_{\mu\nu}$ and iM instead of $\mathcal{X}_\mu,\ \mathcal{P}_\mu,\ \mathcal{J}_{\mu\nu}$ and $\mathcal{M}$ used in the previous chapter as a reminder to the reader that the former are elements of the real Clifford algebra $C\ell(1,3)$. In the algebra (8.5)-(8.13) we have strategically absorbed the i's into the generators because we want to represent the SPHA in terms of the real Clifford algebra $C\ell(1,3)$.

We will call the $X_\mu,\ P_\mu,\ iJ_{\mu\nu}$ and iM above the restricted Clifford generators of the SPHA. Restrictive in the sense that these are particular Clifford algebra definitions of the generators force upon us the conditions that $\alpha_3 = 0$. We adopt the same interpretation of $X_\mu, P_\mu, iJ_{\mu\nu}$ and iM as in the previous chapter. Specifically, X_μ is considered to be the generator of energy momentum translation. We will show in the next chapter that this is a geometrically meaningful definition to attach to X_μ.

We will show in the next section that the generators (8.1)-(8.4) are not uniquely defined but that rotations in the position-momentum plane (that is the X_μ-P_μ plane of the generators) change (8.1)-(8.4), but that the algebra (8.5)-(8.13) remains unchanged.

8.3 General Clifford algebraic representations

The representation of the SPHA given in the previous section was restricted by the condition that the dimensionless parameter β be equal to zero. In the previous chapter it was found that this parameter is however of significant physical importance. It is intrinsically connected to the conformal algebraic limit of SPHA and in this limit β determines how much $\mathcal{X}_\mu$ and $\mathcal{P}_\mu$ mix. It would therefore be encouraging to find a representation in terms of the non-scalar elements of $C\ell(1,3)$ where β need not necessarily be equal to zero.

To find such representations we consider the same approach as was taken in the previous chapter to find a conformal algebraic limit of SPHA. Instead of defining X_μ and P_μ to be mono-vectors and tri-vectors respectively, we define both as general odd vectors in $C\ell(1,3)$. This reinforces the claim made in the previous chapter that events should be characterized not only by their spacetime location but rather by both their location and also by the associated energy momentum.

We start by redefining X_μ and P_μ as

$$X_\mu = a\, e_\mu + b\, ee_\mu, \tag{8.14}$$

$$P_\mu = d\, ee_\mu + c\, e_\mu, \tag{8.15}$$

where a, b, c, d are scalars whose values we will seek to determine. Using this redefined X_μ, the commutator $[X_\mu, X_\nu]$ is equal to

$$[X_\mu, X_\nu] = 2(a^2 + b^2)\, e_{\mu\nu}, \qquad \mu \neq \nu. \tag{8.16}$$

What we require is $[X_\mu, X_\nu] = q\,\alpha_2\, iJ_{\mu\nu}$, which gives the following expression for $iJ_{\mu\nu}$,

$$iJ_{\mu\nu} = \frac{2(a^2+b^2)}{q\,\alpha_2}\, e_{\mu\nu}, \qquad \mu \neq \nu. \tag{8.17}$$

Similarly, for the redefined P_μ,

$$[P_\mu, P_\nu] = 2(c^2+d^2)\, e_{\mu\nu}, \qquad \mu \neq \nu. \tag{8.18}$$

This has to be equal to $[P_\mu, P_\nu] = i\,q\,\alpha_1\, J_{\mu\nu}$ and so we also have

$$iJ_{\mu\nu} = \frac{2(c^2+d^2)}{q\,\alpha_1}\, e_{\mu\nu}, \qquad \mu \neq \nu. \tag{8.19}$$

Consistency in the definition of $iJ_{\mu\nu}$ implies that

$$\frac{a^2+b^2}{\alpha_2} = \frac{c^2+d^2}{\alpha_1}$$

The above transformed definitions for X_μ and P_μ can be written in matrix form as

$$\begin{bmatrix} X_\mu \\ P_\mu \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e_\mu \\ ee_\mu \end{bmatrix} \tag{8.20}$$

and therefore

$$\begin{bmatrix} e_\mu \\ ee_\mu \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} X_\mu \\ P_\mu \end{bmatrix}$$

or

$$e_\mu = \frac{1}{\Delta}(dX_\mu - bP_\mu), \tag{8.21}$$

$$ee_\mu = \frac{1}{\Delta}(-cX_\mu + aP_\mu). \tag{8.22}$$

where $\Delta = ad - bc$ is the determinant of the matrix.

Working out the commutator $[P_\mu, X_\nu]$ with the new transformed expressions for X_μ and P_μ, gives

$$[P_\mu, X_\nu] = 2(ad-bc)\eta_{\mu\nu}\, e + 2(ac+bd)e_{\mu\nu}. \tag{8.23}$$

Equating this with $[P_\mu, X_\nu] = q\eta_{\mu\nu} iM + q\alpha_3 iJ_{\mu\nu}$ yields the following expression for iM and $iJ_{\mu\nu}$,

$$iM = \frac{2(ad-bc)}{q}\, e, \tag{8.24}$$

and

$$iJ_{\mu\nu} = \frac{2(ac+bd)}{q\,\alpha_3}\, e_{\mu\nu}. \tag{8.25}$$

Consistency now requires that

$$\frac{c^2+d^2}{\alpha_1} = \frac{a^2+b^2}{\alpha_2} = \frac{ac+bd}{\alpha_3}.$$

Next consider the commutator $[P_\mu, iM] = \alpha_3 P_\mu - \alpha_1 X_\mu$. We have

$$[P_\mu, iM] = -\frac{4}{q}(ac+bd)P_\mu + \frac{4}{q}(c^2+d^2)X_\mu.$$

By comparing this with $[P_\mu, iM] = \alpha_3 P_\mu - \alpha_1 X_\mu$ gives expressions for α_3 and α_1 in terms of a, b, c, d.

$$\alpha_3 = -\frac{4}{q}(ac+bd), \tag{8.26}$$

$$\alpha_1 = -\frac{4}{q}(c^2+d^2). \tag{8.27}$$

The consistency conditions must again be extended to read

$$\frac{c^2+d^2}{\alpha_1} = \frac{a^2+b^2}{\alpha_2} = \frac{ac+bd}{\alpha_3} = -\frac{q}{4}. \tag{8.28}$$

Similarly, for the commutator $[X_\mu, iM] = \alpha_2 P_\mu - \alpha_3 X_\mu$ we have

$$[X_\mu, iM] = -\frac{4}{q}(a^2+b^2)P_\mu + \frac{4}{q}(ac+bd)X_\mu.$$

Thus

$$\alpha_2 = -\frac{4}{q}(a^2+b^2), \tag{8.29}$$

$$\alpha_3 = -\frac{4}{q}(ac+bd), \tag{8.30}$$

which hold by the above consistency equations.

To satisfy the consistency equations (8.28), we may define

$$a = \sqrt{-\frac{q\,\alpha_2}{4}}\ \cos\theta, \qquad b = \sqrt{-\frac{q\,\alpha_2}{4}}\ \sin\theta,$$

and

$$c = -\sqrt{-\frac{q\,\alpha_1}{4}}\ \sin\phi, \qquad d = \sqrt{-\frac{q\,\alpha_1}{4}}\ \cos\phi.$$

Also since $\dfrac{ac+bd}{\alpha_3} = -\dfrac{q}{4}$, we obtain

$$\sin(\theta-\phi) = \frac{\alpha_3}{\sqrt{\alpha_1\alpha_2}}. \tag{8.31}$$

Rewriting this, we obtain an expression for α_3

$$\alpha_3 = \sqrt{\alpha_1\alpha_2}\sin(\varphi), \qquad \varphi = \theta - \phi. \tag{8.32}$$

Equation (8.32) is an explicit expression for α_3 in terms of α_1, α_2 and and angle parameter $\varphi = \theta - \phi$. Making the substitutions $\alpha_1 = \frac{\hbar}{\ell_C^2}$, $\alpha_2 = \frac{\ell_P}{\hbar}$ and replacing α_3 by β, equation 8.32 can be written as

$$\beta = \frac{\ell_P}{\ell_C}\sin(\varphi). \tag{8.33}$$

One needs to check to see if the above transformation implies a redefinition of the generators $iJ_{\mu\nu}$ and iM or leaves them unchanged. This can easily be checked by substituting a, b, c and d into (8.24) and (8.25) to get

$$\begin{aligned} iM &= \frac{2(ad-bc)}{q}e, \\ &= -\frac{1}{2}\sqrt{\alpha_1\alpha_2}\cos(\theta-\phi)e, \end{aligned} \tag{8.34}$$

$$\begin{aligned} iJ_{\mu\nu} &= \frac{2(a^2+b^2)}{q\alpha_2}e_{\mu\nu}, \\ &= -\frac{1}{2}e_{\mu\nu}. \end{aligned} \tag{8.35}$$

From the above calculations we conclude that the entire SPHA without any restrictions on α_1, α_2 and α_3 other than $\alpha_3^2 - \alpha_1\alpha_2 \neq 0$ is generated by

$$iX_\mu = -\frac{1}{2}\sqrt{q\alpha_2}\left[\cos(\theta)e_\mu + \sin(\theta)ee_\mu\right], \tag{8.36}$$

$$iP_\mu = -\frac{1}{2}\sqrt{q\alpha_1}\left[\cos(\phi)ee_\mu - \sin(\phi)e_\mu\right], \tag{8.37}$$

$$iJ_{\mu\nu} = -\frac{1}{2}e_{\mu\nu}, \tag{8.38}$$

$$iM = -\frac{1}{2}\sqrt{\alpha_1\alpha_2}\cos(\theta-\phi)e. \tag{8.39}$$

We will call (8.36)-(8.39) the Clifford generators of the full SPHA.

It should be noted that the redefinition of iX_μ and iP_μ will determine how the definition of iM is affected. (8.24) and (8.25) give the redefined $iJ_{\mu\nu}$ and iM. iM is affected as shown above whereas $iJ_{\mu\nu}$ are the only generators not affected by the redefinition of X_μ and P_μ.

In this section we have transformed the Clifford generators which give us a representation of the restricted SPHA such that they give us a representation of the full SPHA where α_3 is not necessarily equal to zero. Our approach is in a sense the reverse of the approach taken by Chryssomalakos and Okon [54], who starting with a representation where α_3 is not necessarily zero show that there always exists a representation in the α_1-α_2 plane with α_3 equal to zero by performing a linear redefinition of the generators. Our approach is thus consistent with the approach in [54].

8.4 Physical interpretation of transformation

Chryssomalakos and Okon [54] comment that physicists may frown upon the idea of working with arbitrary linear combinations of momenta and positions. For this reason it is important that we investigate what physical meaning may be attached to such transformations.

The transformation

$$iX_\mu \mapsto -\frac{1}{2}\sqrt{q\alpha_2}\left[\cos\theta e_\mu + \sin\theta e e_\mu\right], \tag{8.40}$$

$$iP_\mu \mapsto -\frac{1}{2}\sqrt{q\alpha_1}\left[\cos\phi e e_\mu - \sin\phi e_\mu\right], \tag{8.41}$$

corresponds to a mixing of the generators of spacetime translation P_μ with the generators of momentum translation X_μ. The transformation looks like a rotation in the position-momentum plane however the transformations are determined by two generally different angles. This means that in general iX_μ and iP_μ are rotated by different amounts. For small θ and ϕ, there is very little mixing of the generators. For $\theta = \phi = \frac{n\pi}{2}$ however, there is an interchange between position and momentum. This is unlikely to correspond to anything physical since the determinant of the transformation will be zero for such a scenario.

When $\theta = \phi$, the parameter α_3 will be equal to zero and $iM = -\frac{1}{2}\sqrt{\alpha_1\alpha_2}e$. The transformation is nothing more than a rotation in the position-momentum plane which leaves the SPHA with $\alpha_3 = 0$ invariant. The Clifford generators found for the SPHA with $\alpha_3 = 0$ in the section 8.2 are thus not unique since rotating the generators in the position-momentum plane leaves the algebra generated invariant. Taking $\theta = \phi = 0$ recovers the original Clifford generators from the previous section.

As noted earlier, the transformation (8.40) and (8.41), implies the subsequent transformation

$$iM \mapsto -\frac{1}{2}\sqrt{\alpha_1\alpha_2}\cos(\theta - \phi)e. \tag{8.42}$$

The physical meaning of iM is uncertain at present. It may be that iM is a dilation. If iM indeed is a dilation then for the case where α_3 is equal to zero, iM has a fixed magnitude, whereas in general the magnitude of iM is dependent on the angle parameter $\varphi = \theta - \phi$. iM then seems to have its maximum magnitude when $\theta = \phi$, *i.e* when $\alpha_3 = 0$. It should be noted that $\cos(\theta - \phi)$ is the determinant of the matrix in the transformation (8.20), which depending on θ and ϕ does not have to be equal to unity.

What is the physical interpretation of this transformation? In a Newtonian mindset we consider time and space to be disjoint and one can determine the absolute time and position of an event. In relativity, time and space can no longer be treated separately but should be considered together in what we call spacetime. It is no longer possible to determine the time and position of an event separately, in a way that different observers will agree.

Similarly, in Newtonian physics, position and momentum are treated separately and thus the above transformation may seem unphysical. In a quantum mechanical framework position X_μ and momentum P_μ cannot be measured independently. The more accurately we know one, the less accurately we know the other as is described by the Heisenberg uncertainty relationship $\Delta X_\mu \Delta P_\mu \geq \frac{\hbar}{2}$. We cannot think about position or momentum without also considering the other. Furthermore, making a measurement of the position of some particle will in itself affect the position. The photon used to measure the particle's position will give the particle some momentum. On the quantum scale therefore a linear combination of position and momentum does make sense and treating position and momentum separately as in Newtonian physics may no longer be desirable.

At the interface of the quantum and relativistic realms we should consider spacetime and energy-momentum not as separate from one another but rather as a single entity. This is in agreement with observations made in the previous chapter. In the previous chapter however the mixing of position and momentum was only considered in the conformal algebraic limit. In this chapter we show that this mixing occurs whenever $\beta \neq 0$, not just in the conformal algebraic limit.

8.5 Summary

In this chapter we have shown that the non-scalar basis elements of the Clifford algebra $C\ell(1,3)$ generate the stabilised Poincaré-Heisenberg algebra under the action of the Lie bracket. The advantage of using the Clifford algebra is that it avoids the complicated mathematics associated with Lie algebraic stability. Stability of a Clifford algebra is assured.

The approach pursued in this chapter gives geometric insight into the nature of the physical space that underlies SPHA. In particular the underlying space is non-commutative, and most likely homogeneous but not continuous. These last two conditions are in conflict with arguments made in reference [5] where it is argued that the physical space remains continuous but loses its homogeneous nature.

It is possible to find both representations when the dimensionless parameter $\alpha_3 = \beta$ is equal to zero and when it is not equal to zero. In the latter case, we were led to consider a mixing of the generators of spacetime and momentum translation. In chapter 7, this mixing was only considered in the conformal algebraic limit and in reference [54] this mixing occurs only when $\beta = 0$.

An expression for β was found in terms of the two length scales ℓ_P and ℓ_C and an angle parameter φ. This expression constrains the value of β, $|\beta| \leq \ell_C/\ell_P$. This constraint suggests the Clifford algebra approach may not give rise to a conformal algebraic limit as in chapter 7.

Chapter 9

On the Geometry of the Conformal Group in Spacetime

9.1 Introduction

The conformal group[1] of most interest to physicists is the conformal group on Minkowski space $\mathbb{R}^{1,3}$. We choose the metric $g = g_{\mu\nu}$ with signature $(+,-,-,-)$ and $c = 1$, and avoid indices by writing $\langle x, y\rangle = x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3$. Although most of the ideas regarding conformal transformations naturally carry over to general $\mathbb{R}^{p,q}$ spaces, the geometric ideas are easier to understand in Minkowski space and we are not burdened with extra notation.

The conformal group can be studied from a number of equivalent viewpoints.

We can compactify $\mathbb{R}^{1,3}$ to a 4-dimensional submanifold of the projective space $P_5(\mathbb{R})$, in which case the identity connected component of the conformal group is isomorphic to $SO(2,4)$. (See e.g. Schottenloher [92] for a detailed outline of this conventional approach whereby non-linear conformal maps are linearised in a larger space.)

Castro and Pavšič [93] have shown the conformal group $SO(2,4)$ emerges from the Clifford algebra of spacetime $C\ell(1,3)$ by pointing out that the conformal group is a subgroup of the Clifford group, but again this leads to an underlying six dimensional space where the extra two components are not easy to identify geometrically.

Hestenes and others [17, 94] developed the idea of the *conformal split* in general $\mathbb{R}^{p,q}$ space and used this to highlight the connection between the conformal group on $\mathbb{R}^{p,q}$ and spin groups which naturally belong to the Clifford algebra $C\ell(p+1, q+1)$.

[1]There is not complete unanimity as to what constitutes the conformal group. Most authors restrict it to a connected component. One advantage of a Clifford algebra approach, is that it naturally leads to a description of the covering group and even allows the inclusion of operators such as inversions, which are not normally included in the conformal group, even though they are conformal transformations.

Lounesto and Latvamaa [95] extended the Clifford algebra $C\ell(p,q)$ to the larger Clifford algebra $C\ell(p+1,q)$ and found simple commutation relations in $C\ell(p+1,q)$ describing the conformal Lie algebra of the conformal group on $\mathbb{R}^{p,q}$. See also Girard [96] for a description of conformal transformations in terms of quaternionic parameters.

In the last two of these approaches, and specialising to Minkowski space, it is recognised that the Clifford algebra $C\ell(1,3)$ is not large enough to accommodate the generators of the conformal Lie algebra on $\mathbb{R}^{1,3}$. In a sense, Lounesto and Latvamaa's description is the simplest since only the time index is increased. But in each of these approaches, the geometric nature of the transformations which make up the conformal group tends to become obscured.

The aim of this paper is to show that the conformal group (more properly, the covering group of the conformal group) can be realised by the action of $C\ell(1,3)$ on the *space* $C\ell(1,3)\bigoplus C\ell(1,3)$. Note that $C\ell(1,3)\bigoplus C\ell(1,3)$ is not an algebra because no multiplication has been defined. It is just a vector space. Although this larger space can be viewed as the vector space of the Clifford algebra $C\ell(2,3)$ (which is the approach Lounesto and Latvamaa take), this is unnecessary. Imposing an algebraic structure tends to obscure the more important geometric ideas and also raises problems of interpretation - e.g., what does the extra generator represent physically or geometrically?

9.2 Conformal transformations

The 10-parameter Poincaré group is the semi-direct product of the 6-parameter Lorentz group with the 4-parameter group of spacetime translations. The Poincaré group may then be enlarged to the conformal group by adding *dilations*

$$x \to \rho x \quad (\rho > 0)$$

as well as *special conformal transformations*

$$x \to \frac{x + \langle x,x\rangle a}{\sigma(x)}, \quad \text{where } \sigma(x) = 1 + 2\langle a,x\rangle + \langle a,a\rangle\langle x,x\rangle$$

which correspond to local scale changes.

The special conformal transformations can be obtained as the product of an inversion

$$I : x \to x^{-1} = \frac{x}{\langle x,x\rangle}$$

followed by a translation and another inversion. To see this, consider an inversion followed by a translation

$$T_a I : x \to \frac{x}{\langle x,x\rangle} + a.$$

Next, consider a second inversion

$$IT_aI : x \to \frac{\frac{x}{\langle x,x\rangle} + a}{\langle \frac{x}{\langle x,x\rangle} + a, \frac{x}{\langle x,x\rangle} + a.\rangle}$$

The denominator may be expanded to give

$$\begin{aligned} \left\langle \frac{x}{\langle x,x\rangle} + a, \frac{x}{\langle x,x\rangle} + a \right\rangle &= \left\langle \frac{x}{\langle x,x\rangle}, \frac{x}{\langle x,x\rangle} \right\rangle + \langle a,a\rangle + 2\left\langle a, \frac{x}{\langle x,x\rangle} \right\rangle, \\ &= \frac{1}{\langle x,x\rangle}\left(1 + \langle a,a\rangle\langle x,x\rangle + 2\langle a,x\rangle\right), \end{aligned}$$

and so we get

$$IT_aI : x \to \frac{x + a\langle x,x\rangle}{(1 + \langle a,a\rangle\langle x,x\rangle + 2\langle a,x\rangle)} = \frac{x + a\langle x,x\rangle}{\sigma(x)}, \tag{9.1}$$

where $\sigma(x) = (1 + \langle a,a\rangle\langle x,x\rangle + 2\langle a,x\rangle)$.

The generators of the (identity component of the) conformal group may be realised as differential operators acting on Minkowski space. The operators corresponding to Lorentz transformations ($M_{\mu\nu}$), translations (P_μ), dilatations (D) and special conformal transformations (K_μ) satisfy the following commutation relations

$$\begin{aligned} [M_{\mu\nu}, M_{\sigma\rho}] &= g_{\mu\rho}M_{\nu\sigma} - g_{\mu\sigma}M_{\nu\rho} + g_{\nu\sigma}M_{\mu\rho} - g_{\nu\rho}M_{\mu\sigma}, \\ [P_\lambda, M_{\mu\nu}] &= g_{\lambda\mu}P_\nu - g_{\lambda\nu}P_\mu, \\ [D, M_{\mu\nu}] &= 0, \\ [K_\lambda, M_{\mu\nu}] &= g_{\lambda\mu}K_\nu - g_{\lambda\nu}K_\mu, \\ [P_\mu, P_\nu] &= 0, \\ [D, P_\mu] &= -P_\mu, \\ [P_\mu, K_\nu] &= 2(M_{\mu\nu} - g_{\mu\nu}D), \\ [D, K_\mu] &= K_\mu, \\ [K_\mu, K_\nu] &= 0. \end{aligned}$$

(Note that there is some divergence between authors. Some require that the generators be Hermitian in which case the imaginary number i makes an occasional appearance in these equations. Since we are dealing with Lie, i.e. anti-symmetric, products it is perhaps more logical to define these generators to be skew-Hermitian. This has the added bonus that only real algebras ever have to be used. For this reason, we follow the definitions in Barut and Raczka [97] and Lounesto [21].)

9.3 The Clifford algebra representations

We regard $C\ell(1,3)$ as acting on the *vector space* $C\ell(1,3) \bigoplus C\ell(1,3)$ by left multiplication. It is then straightforward to verify that the above commutation relations are satisfied by the following operators.

$$\begin{aligned} M_{\mu\nu}(x,y) &= \frac{1}{2}(e_{\mu\nu}x, e_{\mu\nu}y), \\ P_\mu(x,y) &= (e_\mu y, 0), \\ K_\mu(x,y) &= (0, e_\mu x), \\ D(x,y) &= \frac{1}{2}(-x, y), \end{aligned}$$

where $x, y \in C\ell(1,3)$.

Although the inversion operator is not in the identity component, it too has a very natural representation in this context as

$$I(x,y) = (y,x),$$

where in this chapter I is the inversion operator and is not to be confused with the identity operator in other chapters. It now follows that $K_\mu = IP_\mu I$.

Since these operators are defined through the action of the *associative* algebra $C\ell_{1,3}$, they can be expected to have extra algebraic properties.[2] As an example, the translation generators P_μ satisfy the property

$$P_\mu P_\nu = 0,$$

(which trivially implies that $[P_\mu, P_\nu] = 0$). As is shown below in this section, this property is important when we want to show that P_μ generates a translation in the direction e_μ.

The transformations that arise from these generators are now easy to describe geometrically.

9.3.1 Lorentz transformations

It is well known that the elements $M_{\mu\nu}$ generate (the proper orthochronous) Lorentz transformations on $\mathbb{R}^{1,3}$. The Lorentz transformations are generated by

$$\exp(tM_{\mu\nu})(x,y) = \left(1 + tM_{\mu\nu} + \frac{t^2 M_{\mu\nu}^2}{2!} + ...\right)(x,y).$$

[2] As a general rule, if a Lie algebra structure is imposed on an associative algebra via $[A,B] = AB - BA$, some properties of AB may be lost

$M_{\mu\nu}^2$ is equal to either $+\frac{1}{4}$ or $-\frac{1}{4}$. Assuming first that $M_{\mu\nu}^2 = -\frac{1}{4}$, we obtain

$$\begin{aligned}\exp(tM_{\mu\nu})(x,y) &= \left(1 - \frac{(t/2)^2}{2!} + ...\right)(x,y) + \left(t/2 - \frac{(t/2)^3}{3!} - ...\right)(e_{\mu\nu}x, e_{\mu\nu}y),\\ &= (\cos(t/2)x + e_{\mu\nu}\sin(t/2)x,\ \cos(t/2)y + e_{\mu\nu}\sin(t/2)y)\,,\end{aligned}$$

and so $M_{\mu\nu}$ generates the rotations. On the other hand, when $M_{\mu\nu}^2 = +\frac{1}{4}$, we have

$$\begin{aligned}\exp(tM_{\mu\nu})(x,y) &= \left(1 + \frac{(t/2)^2}{2!} + ...\right)(x,y) + \left(t/2 + \frac{(t/2)^3}{3!} + ...\right)(e_{\mu\nu}x, e_{\mu\nu}y),\\ &= (\cosh(t/2)x + e_{\mu\nu}\sinh(t/2)x, \cosh(t/2)y + e_{\mu\nu}\sinh(t/2)y)\,,\end{aligned}$$

that is, the boosts.

In the Clifford algebra setting we could define $M_{\mu\nu} = \dfrac{e_{\mu\nu}}{2}$. Then, for example, a boost in the direction $n = (n_1, n_2, n_3)$, with velocity $v = \tanh\phi$ (remember that we are using units with $c = 1$), may be represented as

$$x \to x' = axa^{-1},$$

where $a = \exp(\phi n)$ and $n = n_1e_{01} + n_2e_{02} + n_3e_{03}$. Similarly spatial rotations through an angle θ, are of the same form but now $a = \exp(\theta ne)$ with n describing the axis of rotation and $e = e_{0123}$.

9.3.2 Translations

Consider now the generator P_μ defined by

$$P_\mu(x,y) = (e_\mu y, 0).$$

Clearly $P_\mu^2 = 0$ so that

$$\begin{aligned}\exp(tP_\mu)\,(x,y) &= (1 + tP_\mu)(x,y),\\ &= (x,y) + tP_\mu(x,y),\\ &= (x,y) + t(e_\mu y, 0),\\ &= (x + te_\mu y, y).\end{aligned}$$

More generally, if $a = (a^\mu e_\mu)$ is a vector in $\mathbb{R}^{1,3}$, then

$$\exp(ta^\mu P_\mu)\,(x,y) = (x + ta^\mu e_\mu y, y),$$

and in particular

$$\exp(ta^\mu P_\mu)\,(x,1) = (x + ta, 1),$$

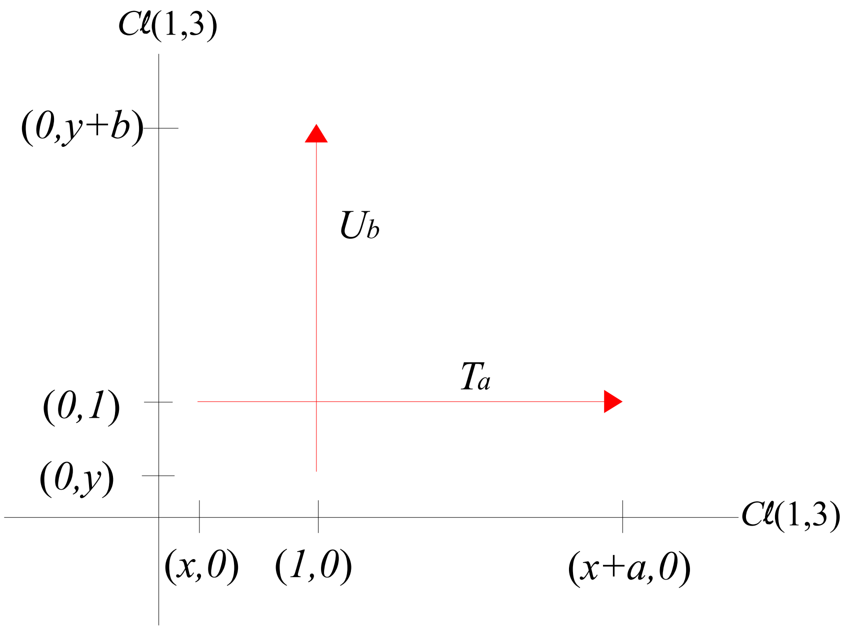

Figure 9.1: Geometric representation of the translations and special conformal transformations on the space $C\ell_{1,3} \oplus C\ell_{1,3}$. The generators T_a and U_b act on the hyperspaces $\mathbb{R}^{1,3} \bigoplus (1)$ and $(1) \bigoplus \mathbb{R}^{1,3}$ respectively.

so that $P_a = a^\mu P_\mu$ generates the translation operation $T_a : x \rightarrow x + a$ in $\mathbb{R}^{1,3}$ when this space is identified with the hyperplane $\mathbb{R}^{1,3} \bigoplus (1)$ in $C\ell(1,3) \bigoplus C\ell(1,3)$, see Figure 9.1

It might also be worth pointing out, that there is no *algebraic* reason why we should only consider translations in the direction of a 1-vector. If u is *any* element of $C\ell(1,3)$, we can define an operator P_u by

$$P_u(x, y) = (uy, 0),$$

and this generates a translation in $C\ell(1,3) \bigoplus C\ell(1,3)$. This then leads to a generalization of the conformal group and it would be interesting to characterize this extended group further. It should be noted however that in general $u^2 \neq 0$ and therefore in general

$$\exp(tP_\mu)\,(x, y) \neq (x + tuy, y).$$

9.3.3 Special conformal transformations

The generator K_μ behaves much like P_μ, but on the second component space of $C\ell(1,3) \bigoplus C\ell(1,3)$. It too generates a translation $U_b : y \rightarrow y + b$ in $\mathbb{R}^{1,3}$ when the space in identified with the hyperplane $(1) \bigoplus \mathbb{R}^{1,3}$, this time of the form

$$\exp(ta^\mu K_\mu)\,(1, y) = (1, y + ta).$$

This illustrates an advantage of our approach. The special conformal transformations act in an entirely similar way to translations, but on the second component subspace rather than the first. In that sense, they are no more non-linear than translations, see Figure 9.1.

Again we can generalize special conformal transformations to operators K_u defined by

$$K_u(x, y) = (0, ux).$$

9.3.4 Dilatations

The operator D defined by $D(x, y) = \frac{1}{2}(-x, y)$ generates the transformations

$$\begin{aligned}
\exp(tD)(x, y) &= \left(1 + tD + \frac{t^2 D^2}{2!} + \frac{t^3 D^3}{3!} + ...\right)(x, y), \\
&= \left(1 + \frac{(t/2)^2}{2!} + ...\right)(x, y) + \left(t + \frac{(t/2)^3}{3!} + ...\right)(-x, y), \\
&= \left(\mathrm{e}^{-\frac{t}{2}} x,\ \mathrm{e}^{\frac{t}{2}} y\right).
\end{aligned}$$

These transformations can, in the special cases where either x or y is 0, represent dilations of 1-vectors in $\mathbb{R}^{1,3}$. Again, there is no algebraic reason why dilatations cannot be considered on all of $C\ell(1, 3)$ or in fact, on all of $C\ell(1, 3) \bigoplus C\ell(1, 3)$.

9.3.5 Inversions

Although inversions are not part of the connected component of the conformal group (and hence do not appear in the conformal Lie algebra), they are conformal transformations which in our picture, interchange the two component subspaces and thus provide a link between special conformal transformations and translations, see Figure 9.2.

9.4 Summary

In this chapter we have proposed an alternative linearization procedure for the conformal group of $\mathbb{R}^{p,q}$. To achieve linearization, we let the conformal transformations act on two copies of the associated Clifford algebra instead of the standard procedure which involves compactifying $\mathbb{R}^{p,q}$,(so that the conformal transformations may be represented by linear transformations in $\mathbb{R}^{p+1,q+1}$). In particular we have considered the conformal transformations of Minkowski space $\mathbb{R}^{1,3}$ to highlight the geometrical advantages provided by this Clifford algebra approach.

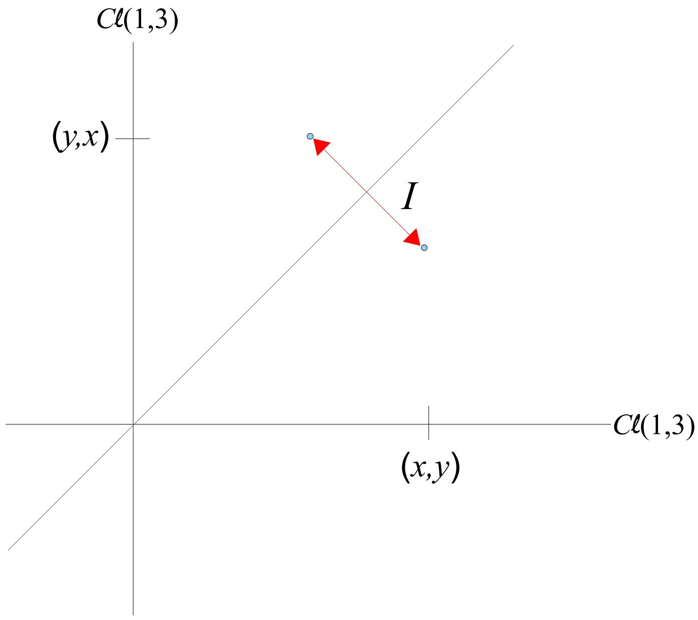

Figure 9.2: Geometric representation of an inversion. An inversion corresponds to a reflection about the 'line' (x, x).

Representing the conformal algebra in $\mathbb{R}^{1,3}$ in terms of $C\ell(1,3)$ rather than some larger Clifford algebra preserves, and in fact emphasizes, the geometric nature of conformal transformations.

Chapter 10

Conclusion

10.1 Summary of results

In this thesis we have motivated the use of the Clifford algebra $C\ell(1,3)$ to formulate physical theories in spacetime. In particular, we have shown how Clifford algebras arise from the underlying geometry of a space. Once we derived the algebra $C\ell(1,3)$ from the geometry of spacetime, the remaining chapters of the thesis served to highlight how the algebra is a versatile and useful tool for physicists. A theory of electromagnetism was formulated in the algebra and it was also shown that the algebra is consistent with an SPHA-based theory of quantum cosmology.

In chapter 2 we summarised an attempt made in [8] to show how the spacetime Clifford algebra $C\ell(1,3)$ can be derived from the underlying geometry of spacetime.

Starting with two dimensional geometry, it was shown that in addition to a scalar and two orthonormal basis vectors, another mathematical object called a bi-vector is obtained by taking the geometric product of the two basis vectors. This bi-vector turned out, under left multiplication, to induce a $+\frac{\pi}{2}$ rotation in the plane of the two orthonormal basis vectors.

In three spatial dimensions, rotations are not commutative. The eight dimensional Clifford algebra $C\ell(0,3)$ contains three bi-vectors that do not commute. These bi-vectors provide a tool for describing rotations geometrically in a plane instead of about an axis of rotation. The choice of metric (and consequently, the choice of Clifford algebra) is important. Only the anti-Euclidean metric preserves handedness.

We generalised to spacetime and found that the sixteen dimensional Clifford algebra $C\ell(1,3)$ is the correct algebra for formulating spacetime physics.

There is a prevailing misidentification between the axial vectors and the polar vectors in 3-space that on the geometric level equates planes with vectors perpendicular to the planes. The Clifford algebra keeps the distinction between vectors and planes (bi-vectors) very clear, the latter being obtained via the geometric product of two vectors. This geometric product is defined for any dimension unlike the cross product.

It was shown, in contradiction to popular belief, that the choice of metric is important and not merely a matter of taste. Only the anti-Euclidean metric with signature $(-,-,-)$ and Lorentzian metric with signature $(+,-,-,-)$ allow proper treatments of rotations.

Matrices are a natural and useful way of studying various properties of (Clifford) algebras. In chapter 3 the matrix representations for some Clifford algebras were given. The spacetime algebra $C\ell(1,3)$ can be represented in terms of 2×2 matrices with quaternions as entries. This representation highlights the link between the quaternions and the bi-vectors e_{ij}, the Pauli spin matrices and the bi-vectors e_{0i}. Rotations can be given a proper treatment in this algebra. One down side of working with matrices is that the matrix representations sometimes obscure the geometry that is so transparent in Clifford algebras.

The spacetime Clifford algebra $C\ell(1,3)$ is not a division algebra. This has led some to suggest that the algebra is therefore not a suitable mathematical structure to model the physical behaviour of nature. Indeed the existence of inverses is very important. In chapter , we gave two reasons why knowing which multivectors are and which are not invertible is important physically. The areas of the algebra where division is not defined correspond to the limiting cases of physical interest.

We defined a new group called the extended Clifford group which does not preserve grade but parity. The Clifford group is a subgroup of the extended subgroup. It was found that all the elements of this group are precisely those elements of the spacetime algebra that are invertible and homogeneous.

In the same chapter we also confirmed some of the results found in [10]: that the areas of the algebra where division is not defined correspond exactly to the limiting cases of physical interest. Therefore, the behaviour of the algebra is in harmony with the behaviour of nature. This led us to reason that the breakdown of division in certain areas of the algebra should not be considered a weakness of the algebra but that it is in fact a strength of the algebra.

The extended Clifford group of spacetime is less restrictive than the Clifford algebra and may be more appropriate to describe the symmetries of the generalised Maxwell equations and the stabilised Poincaré-Heisenberg algebra. It was shown that any element of the extended Clifford group of spacetime can be written as the product of an element of the Lorentz group $\mathrm{SL}(2,\mathbb{C})$ and the unitary group $\mathrm{U}(1)$.

In chapter 5 we have shown that the Lorentz force law follows from supplementing Maxwell's equations by the Newtonian or Hamiltonian principle that force is related to the gradient of energy, $\mathbf{F} = -\nabla U$.

The approach proposed is different in several respects to the common approach: It is symmetric with respect to the sources of the fields; it is Lorentz invariant and uses the retarded fields for all the fields contributing to the total field at each point in space; and the interaction energy (and hence the force experienced by each of the sources) is not determined simply by the electromagnetic field at the position of one of the charges but rather from the sum of the electromagnetic fields due to all charges throughout all space. While this last point may be seen as a disadvantage, we are able to derive the Lorentz force law from Maxwell's equations, together with Noether's theorem.

While many authors, and essentially all undergraduate texts, follow the historical development and derive Maxwell's equations from the force laws, it is well known that Maxwell's equations follow from considerations of possible 4-vector fields in a Lorentz relativistic model.

There are a number of implications of our approach, and there needs to be a re-interpretation of the electromagnetic field. In particular, electromagnetic fields are usually assumed not to interact with each other. We have shown that by laying aside this assumption, one may successfully calculate the electromagnetic forces between charged particles.

In chapter 6 we constructed a theory of electromagnetism using the algebra $C\ell(1,3)$. One common way of writing Maxwell's equations is using the vector notation of Heaviside and Gibbs. The fields in this notation however do not transform correctly because they do not form 4-vectors. In an explicitly covariant formulation of Maxwell's equations, the electric and magnetic fields are contained together inside the antisymmetric second rank Faraday tensor. Such an approach allows one to write Maxwell's equations as just two equations

$$\partial_\mu \widetilde{F}^{\mu\nu} = 0, \qquad \partial_\mu F^{\mu\nu} = J^\nu.$$

In $C\ell(1,3)$, Maxwell's equations are written as a single geometric equation

$$dF = -J.$$

Maxwell's equations can also be written in terms of a mono-vector potential A. Assuming the Lorenz condition (the scalar part of dA is zero), Maxwell's equations are

$$d^2 A = J.$$

The Proca equations are written in much the same way with the exception of an extra mass term which couples to the potential A

$$dF = -m^2 A + J.$$

By considering the Klein-Gordon equation for a spin one field and following the same procedure as one does to find the Dirac equation for a spin one-half field, one arrives not at the Maxwell equations as would be expected but instead at a more general set of equations called the generalised Maxwell equations that includes extra scalar fields.

Starting with a more general potential P, consisting of mono-vector plus tri-vector, and not imposing any gauge conditions on this potential, the source free generalised Maxwell equation can be derived in $C\ell(1,3)$ as

$$d^2 P = 0. \tag{10.1}$$

The energy conservation law for the generalised Maxwell equations differs from the energy conservation law for Maxwell's equations due to the presence of two extra fields.

In chapter 7 we motivated a new candidate for the algebra that may underlie a physically viable and consistent theory of quantum cosmology. Besides yielding an algebraic unification of the extreme microscopic and cosmological scales, it generalises the notion of conformal symmetry. The modifications to the Heisenberg algebra at the present epoch of the universe are negligibly small; but when ℓ_C and ℓ_P are of the same order, the modifications are significant resulting in the breakdown of the continuous and homogeneous nature of physical space that underlies the Heisenberg algebra. An important aspect of the SPHA-based theory of quantum cosmology is that it inevitably provides inhomogeneities or non-continuity in the underlying physical space. Furthermore, in this class of theories one must expect a strong polarisation and spin dependence of various cosmologically induced quantum effects.

There is a limit in which SPHA reduces to the conformal algebra. This limit describes how the present day Poincaré-algebraic description relates to the conformal-algebraic description of the universe in the past. It was shown that the dimensionless parameter β is closely related to the geometry of the underlying physical space. The physical space that underlies the conformal algebra combines the notions of spacetime and energy momentum through the mixing of $\mathcal{X}_\mu$ and $\mathcal{P}_\mu$. The extent of this interplay is governed by the value of β.

We have shown in chapter 8 that the non-scalar basis elements of $C\ell(1,3)$ generate the SPHA under the Lie bracket. We concluded from this that $C\ell(1,3)$ is consistent with a SPHA-based theory of quantum cosmology.

The Clifford algebraic representations introduce a mixing of the position and momenta operators, not only in the conformal algebraic limit but for the SPHA with non vanishing β in general. This result suggests that the next evolutionary step toward a theory of physics at the interface of the quantum and cosmological realms might be to depart from working in spacetime alone but instead work with spacetime and energy-momentum as a single entity.

It was shown that the parameter β may be expressed in terms of the two length scales ℓ_P and ℓ_C and and angle parameter. This may further help understand the role of β and contribute to understanding how β affects various quantum relativistic notions.

In the final chapter of this thesis we have put forth an alternative linearization procedure for the conformal group of $\mathbb{R}^{p,q}$. To achieve linearization, we let the conformal transformations act on two copies of the associated Clifford algebra instead of the standard procedure which involves compactifying $\mathbb{R}^{p,q}$, (so that the conformal transformations may be represented by linear transformations in $\mathbb{R}^{p+1,q+1}$). In particular we have considered the conformal transformations of Minkowski space $\mathbb{R}^{1,3}$ to highlight the geometrical advantages provided by this Clifford algebra approach.

Representing the conformal algebra in $\mathbb{R}^{1,3}$ in terms of $C\ell(1,3)$ rather than some larger Clifford algebra, preserves and in fact emphasizes the geometric nature of conformal transformations.

10.2 Open problems

- To represent the generalised Maxwell equation in $C\ell(1,3)$ we were led to consider a general odd vector potential $\alpha + \beta e$ rather than a mono-vector potential α as for the ordinary Maxwell equations. Similarly, Clifford algebraic representations of the SPHA required us to write both the generators X_μ and P_μ as general odd vectors instead of mono-vectors and tri-vectors respectively. It would be interesting to investigate what this means at the geometric level and if there is some underlying principle involved.

- The extended Clifford group preserves parity but not grade. What is the importance of this group with respect to the previous point? Does this group contain the symmetries of the SPHA, and if so what new physics can then be found in an SPHA-based approach to quantum gravity?

- The SPHA admits three Casimir invariants which differ from the usual two Casimir invariants of the Poincaré algebra, mass and spin, which we use to label quantum mechanical states. The existence of a third Casimir invariant means quantum gravitational states should be labeled by mass, spin and a third quantity. It would be interesting to find what this third quantity is and what the extra terms in the Casimir operators tell us physically.

- The conformal limit of the SPHA suggests a significant mixing of the position and momenta. We also found this was required for a Clifford representation with $\alpha_3 \neq 0$. The amount of mixing is determined by the value of α_3. Is there some way to determine the value of α_3 for the present cosmic epoch and perhaps devise an experiment that could measure it?

- Our derivation of the Lorentz law from the total interaction energy in the system led us to a re-interpretation of electromagnetic fields as interacting with one another. Perhaps the source terms in Maxwell's equations are themselves purely electromagnetic. It would be interesting to investigate this and to see if the extra terms in the generalised Maxwell equations can play the role of sources as has been suggested by Lee [50].

- It has been suggested by Williamson and van der Mark [11] that the electron is a photon with toroidal topology and therefore a purely electromagnetic entity. Our re-interpretation of the electromagnetic field may aid in constructing such a theory of the electron and other particles.

- In chapter 7 we argued that for a universe with conformal symmetry, a general relativistic description of physical reality will require modification. More work needs to be done to determine possible modifications and how these modifications effect various cosmological models.

- In chapter 7 it was shown that there exists a conformal-algebraic limit to the SPHA. This limit does not seem to exist in the Clifford algebraic representations of the SPHA. Why this is the case is unsure at present.

- In the conformal-algebraic limit, the dimensionless parameter β takes on a value whose magnitude is greater than unity. The expression obtained for β in chapter 8 however is constrained in value by $-\frac{\ell_P}{\ell C} \leq \beta \leq \frac{\ell_P}{\ell C}$.

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